BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE.

EDITED BY

T. S. FISKE, F. N. COLE, A. ZIWET,
F. MORLEY, E. O. LOVETT.

VOL. V

OCTOBER 1898 TO JULY 1899.

PUBLISHED FOR THE SOCIETY
BY
THE MACMILLAN COMPANY,
LANCASTER, PA., AND NEW YORK,
1899.

2ND SERIES, VOL. V. NO. 1

OCTOBER, 1898.

Whole No. 71.

CONTENTS.

PAGE.

The Fifth Summer Meeting of the American Mathematical Society. By Professor F. N. COLE, 1

Note on the Generalization of Poincaré and Goursat's Proof of a Theorem of Weierstrass's. By Professor W. F. OSGOOD, 14

Supplementary Note on a Single-Valued Function with a Natural Boundary, whose Inverse is also Single-Valued. By Professor W. F. OSGOOD, 17

Note on the Periodic Developments of the Equation of the Center and of the Logarithm of the Radius Vector. By Professor ALEXANDER S. CHESSIN, 18

The Theorems of Oscillation of Sturm and Klein. By Professor MAXIME BÔCHER, 22

Notes, 44

New Publications, 51

2ND SERIES, VOL. V. NO. 2

NOVEMBER, 1898.

Whole No. 72.

CONTENTS.

PAGE.

The Cambridge Colloquium. By Professor H. S. WHITE, 57

Selected Topics in the General Theory of Functions. By Professor W. F. OSGOOD, 59

The Fiftieth Anniversary Meeting of the American Association for the Advancement of Science. By Professor JAMES MCMAHON, 87

Notes, 105

New Publications, 110

2ND SERIES, VOL. V. NO. 3.

DECEMBER, 1898.

Whole No. 73.

CONTENTS.

PAGE.

The October Meeting of the American Mathematical Society. By Professor F. N. COLE, 113

Concerning a Linear Homogeneous Group in $C_{m,q}$ Variables Isomorphic to the General Linear Homogeneous Group in m Variables. By Dr. L. E. DICKSON, 120

A Second Locus Connected with a System of Coaxial Circles. By Professor THOMAS F. HOLGATE, 135

Reciprocal Transformations of Projective Coördinates and the Theorems of Ceva and Menelaos. By Professor ARNOLD EMCH, 143

Notes, 150

New Publications, 157

2ND SERIES, VOL. V. NO. 4.

JANUARY, 1899.

Whole No. 74.

CONTENTS.

PAGE.

Report on the Theory of Projective Invariants: The Chief Contributions of a Decade. By Professor H. S. WHITE, 161

Reye's Geometrie der Lage. By Professor CHARLOTTE ANGAS SCOTT, 175

Burkhardt's Theory of Functions. By Professor MAXIME BÔCHER, 181

Darboux's Orthogonal Systems. By Professor EDGAR ODELL LOVETT, 185

The New Mathematical Encyclopedia. By Professor JAMES PIERPONT, 202

Errata, 207

Notes, 208

New Publications, 212

2ND SERIES, VOL. V. NO. 5.

FEBRUARY, 1899.

Whole No. 75.

CONTENTS.

PAGE.

The Fifth Annual Meeting of the American Mathematical Society. By Professor F. N. COLE, 217
The December Meeting of the Chicago Section. By Professor THOMAS F. HOLGATE, 222
Report on Recent Progress in the Theory of Groups of a Finite Order. By Dr. G. A. MILLER, 227
Note on Burnside's Theory of Groups. By Dr. G. A. MILLER, 249
On a Regular Configuration of Ten Line Pairs Conjugate as to a Quadric. By Professor F. MORLEY, 252
Bessel's Functions. By Dr. VIRGIL SNYDER, 253
Shorter Notices, 259
Notes, 265
New Publications, 272

2ND SERIES, VOL. V. NO. 6.

MARCH, 1899.

Whole No. 76.

CONTENTS.

PAGE.

On Singular Points of Linear Differential Equations with Real Coefficients. By Professor MAXIME BÔCHER, 275
The Hessian of the Cubic Surface. By Dr. J. I. HUTCHINSON, 282
On the Simple Isomorphisms of a Hamiltonian Group to Itself. By Dr. G. A. MILLER, 292
Galois's Collected Works. By Professor JAMES PIERPONT, 296
Three Memoirs on Geometry. By Professor EDGAR ODELL LOVETT, 300
Stahl's Abelian Functions. By Professor VIRGIL SNYDER, 308
Calculus of Finite Differences. By Professor D. A. MURRAY, 313
Notes, 316
New Publications, 321

2ND SERIES, VOL. V. NO. 7

APRIL, 1899.

Whole No. 77.

CONTENTS.

PAGE.

The February Meeting of the American Mathematical Society. By Professor F. N. COLE, 325

Determinants of Quaternion. By Professor JAMES MILLS PEIRCE, 335

The Largest Linear Homogeneous Group with an Invariant Pfaffian. By Dr. L. E. DICKSON, 338

Asymptotic Lines on Ruled Surfaces Having Two Recti-linear Directrices. By Dr. VIRGIL SNYDER, 343

Wilson's Graphics. By Dr. J. B. CHITTENDEN, 353

Pascal's Repertorium of Higher Mathematics. By Professor EDGAR ODELL LOVETT, 357

D'Ocagne's Descriptive and Infinitesimal Geometry. By Professor EDGAR ODELL LOVETT, 362

Sophus Lie. By Professor GASTON DARBOUX. Translated by Professor EDGAR ODELL LOVETT, 367

Notes, 370

New Publications, 374

2ND SERIES, VOL. V. NO. 8.

MAY, 1899.

Whole No. 78.

CONTENTS.

PAGE.

The April Meeting of the Chicago Section. By Professor THOMAS F. HOLGATE, 379

An Elementary Proof that Bessel's Functions of the Zeroth Order Have an Infinite Number of Real Roots. By Professor MAXIME BÔCHER, 385

A Generalization of Appell's Factorial Functions. By Dr. E. J. WILCZYNSKI, 388

On the Arithmetization of Mathematics. By Professor JAMES PIERPONT,

Two Books on Tides. By Professor ERNEST W. BROWN, 406

Notes, 413

New Publications, 419

2ND SERIES, VOL. V. No. 9

JUNE, 1899.

Whole No. 79.

CONTENTS.

PAGE.

The April Meeting of the American Mathematical Society. By Professor F. N. COLE, 423
Surfaces of Revolution in the Theory of Lamé's Products. By Dr. F. H. SAFFORD, 431
Picard's Algebraic Functions of Two Variables. By ARTHUR BERRY, M.A., 438
Note on Page's Ordinary Differential Equations. By Dr. L. E. DICKSON, 451
Tannery's Arithmetic. By Professor JAMES PIERPONT. 455
Notes, 458
New Publications, 461

2ND SERIES, VOL. V. No. 10.

JULY, 1899.

Whole No. 80.

CONTENTS.

PAGE.

The Asymptotic Lines of the Kummer Surface. By Dr. J. I. HUTCHINSON, 465
On a Definitive Property of the Covanant. By Mr. C. J. KEYSER, 468
The Known Finite Simple Groups. By Professor L. E. DICKSON, 470
The Geometry of Movement. By. Professor F. MORLEY, 476
The New Edition of Weber's Algebra. By Professor JAMES PIERPONT, 480
Shorter Notices, 483
On Elliptic Functions. By Professor JAMES PIERPONT, 490
Notes, 493
New Publications, 496
Eight Annual List of Published Papers, 499
Index, 502

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Fifth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at the Massachusetts Institute of Technology, Boston, Mass., on Friday and Saturday, August 19 and 20, 1898. The Society renewed this year its usual policy of affiliation, so far as concerns the Summer Meeting, with the American Association for the Advancement of Science. This arrangement presents several advantages and involves no sacrifice of independence on either side. A more satisfactory and efficient condition would result if the Association should at some time decide to resolve itself into a federation of scientific societies, a policy which has a strong support. Meanwhile very cordial relations exist between the Society and Section A of the Association. Members of either body were free to attend and participate in the scientific proceedings of the other. The date of the Society's meeting was so chosen as to avoid conflict with that of the Section; and to accommodate those members of the Society who were in attendance on the Colloquium the Section set apart a special day for the reading of its chief mathematical papers.

In attendance and in number of papers presented the Fifth Summer Meeting surpassed any of its predecessors. The total attendance at the three sessions exceeded sixty-five, and included the following fifty-three members of the Society:

Professor A. L. Baker, Dr. C. C. Barnum, Professor M. Bôcher, Professor W. E. Byerly, Professor C. H. Chandler, Professor A. S. Chessin, Dr. J. B. Chittenden, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor C. L. Doolittle, Professor L. W. Dowling, Professor H. T. Eddy, Professor T. S. Fiske, Dr. A. B. Frizell, Dr. J. W. Glover, Miss Ida Griffiths, Mr. G. H. Hallett, Mr. H.

E. Hawkes, Professor Ellen Hayes, Dr. J. I. Hutchinson, Professor H. Jacoby, Mr. C. J. Keyser, Professor P. Ladue, Professor P. A. Lambert, Professor G. Lanza, Dr. G. H. Ling, Professor F. H. Loud, Mr. F. M. McGaw, Professor J. McMahon, Dr. A. Macfarlane, Professor M. Merriman, Professor W. H. Metzler, Professor E. H. Moore, Dr. D. A. Murray, Professor J. C. Nagle, Professor S. Newcomb, Professor W. F. Osgood, Professor B. O. Peirce, Mr. D. L. Pettegrew, Professor J. Pierpont, Professor C. Puryear, Professor P. F. Smith, Dr. V. Snyder, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor C. B. Williams, Miss E. C. Williams, Professor F. S. Woods, Professor R. S. Woodward, and Professor T. W. D. Worthen.

The opening session began at 10 a. m. on Friday. The President, Professor Simon Newcomb, occupied the chair and delivered a brief address on the remarkable growth of mathematical science in this country. Having only strictly scientific business before it, the meeting was able to complete the lengthy programme in three sessions, finishing at noon on Saturday. At the last two sessions Vice-Presidents R. S. Woodward and E. H. Moore presided. The Council announced the election of the following persons to membership in the Society: Mr. H. E. Hawkes, Yale University; Dr. F. H. Safford, Harvard University; Dr. F. Schlesinger, Yerkes Observatory; Dr. J. Westlund, Yale University. Six applications for membership were received. At the meeting of the Council on Friday evening, a committee consisting of Professors T. S. Fiske, Simon Newcomb, E. H. Moore, M. Bôcher, and J. Pierpont was appointed to consider the question of securing improved facilities for the publication of original mathematical papers in this country.

The following papers were read at the meeting:

(1) Dr. E. M. BLAKE: "On the ruled surfaces generated by the plane movements whose centrodes are congruent conics tangent at homologous points." (Illustrated by models.)

(2) Professor T. F. HOLGATE: "A second locus connected with a system of coaxial circles."

(3) Dr. J. I. HUTCHINSON: "On the Hessian of the cubic surface."

(4) Dr. VIRGIL SNYDER: "Asymptotic lines on cubic scrolls."

(5) Professor ALEXANDER CHESSIN: "Relative motion considered as disturbed absolute motion."

(6) Professor A. L. BAKER: "Fundamental algebraic operations."

(7) Professor ALEXANDER CHESSIN: "On the development of the perturbative function in terms of the mean anomalies."

(8) Professor E. O. LOVETT: "Note on the differential invariants of a system of $m+1$ points by projective transformation."

(9) Professor W. F. OSGOOD: "Note on the extension of the Poincaré-Goursat proof of a theorem of Weierstrass's."

(10) Professor W. F. OSGOOD: "Supplementary note on a single-valued function with a natural boundary, whose inverse is also single-valued."

(11) Professor MAXIME BÔCHER: "The theorems of oscillation of Sturm and Klein."

(12) Professor A. L. BAKER: "Space concepts in mathematics."

(13) Dr. T. P. HALL: "An algebra of space."

(14) Professor E. H. MOORE: "The subgroups of the generalized modular group."

(15) Professor L. L. CONANT: "An application of the theory of substitutions."

(16) Dr. J. H. BOYD: "A method for finding an approximate integral for any differential equation of the second order."

(17) Dr. H. F. STECKER: "Non-euclidean cubics."

(18) Dr. G. A. MILLER: "On the simple isomorphisms of a Hamiltonian group to itself."

(19) Dr. L. E. DICKSON: "A new triply-infinite system of simple groups obtained by a twofold generalization of Jordan's first hypoabelian group."

(20) Dr. L. E. DICKSON: "Construction of a linear homogeneous group in C_q^m variables, isomorphic to any given linear homogeneous group in m variables."

(21) Dr. JACOB WESTLUND: "On a class of equations of transformation."

(22) Professor F. MORLEY: "A generalization of Desargues's theorem."

(23) Dr. E. L. STABLER: "A rule for finding the day of the week corresponding to a given date."

(24) Dr. ARTEMAS MARTIN: "Evolution by logarithms."

(25) Dr. ARTEMAS MARTIN: "A method of finding, without tables, the number corresponding to a given logarithm —II."

Dr. Boyd was introduced by Professor C. B. Williams. Dr. Blake's and Professor Morley's papers were read by Professor Fiske, Professor Holgate's by Professor White, Dr. Stecker's by Professor Woods, and Dr. Westlund's by Pro-

fessor Pierpont. In the absence of the authors the papers of Professor Lovett, Dr. Hall, Dr. Miller, Dr. Dickson, Dr. Stabler, and Dr. Martin were read by title.

Abstracts of those papers which are not intended for publication in the BULLETIN are given below.

Dr. Blake's paper is intended for publication in the *Annals of Mathematics*. The movements considered are defined as follows: upon a plane α' containing a conic C' moves a coincident plane α containing a congruent conic C, in such a manner that the two conics are always tangent at homologous points. The locus of a point carried by α is a unicursal curve of the fourth order when C and C' are central conics and a unicursal curve of the third order when they are parabolas. The properties of these curves taken from the writings of Roberts, Cayley, and others are given in brief. The locus of a straight line carried by α is a quartic scroll with a nodal circle at infinity and a nodal straight line when C and C' are central conics; and a cubic scroll with a nodal straight line when they are parabolas. These surfaces admit of considerable variety with regard to the reality and aggregation of their pinch points. They are described under thirteen types. The study of them from the point of view of their mechanical generation is believed to be new.

Professor Chessin's papers will be published in the *American Journal of Mathematics*. The following is an outline of the first paper: Let a space R move within a space A and call these spaces respectively the *relative* and the *absolute* spaces. The motion of any system S may be considered as taking place either in the space A or in the space R. In the first case the motion of S is called *absolute*, in the second *relative*. The motion of R within A is called space motion. If there were no space motion the motion of S in R would be identical with its motion in A. We may therefore consider the space motion as a *perturbation of the absolute motion*, the latter being the *undisturbed* motion, while the relative motion will become the *disturbed* motion. The object of the paper presented was to give an expression for the perturbative function which enables us to pass from the absolute to the relative motion in a way similar to that followed in the general theory of perturbations. The expressions obtained for the perturbative function Ω show that it is composed of two parts, one independent of the equations of constraint to which S

may be subject and the other directly depending on these equations. It was particularly to give expressions for the latter part of Ω that the present investigation was made, expressions for the other part having been obtained before.

In the *Astronomical Journal* (1894, Nos. 326 and 332) Professor Chessin has given a method which greatly reduces the work of computation and simplifies the development of the perturbative function in terms of the eccentric anomalies as given by Professor Simon Newcomb in the Astronomical Papers, vol. 3, Part I (see also Tisserand, Mécanique Céleste, vol. 4, p. 313). In his second paper before the Society a similar method is given by Professor Chessin for the development of the perturbative function in terms of the mean anomalies. The method applies as well to the development of Le Verrier, as to that given by Professor Newcomb (Astronomical Papers, vol. 5, Part I), although it is here applied only to the latter development. It is estimated that the actual work of computation is thus reduced to about one-fifth of what it would be if the current methods were used.

Professor Lovett's paper will be published in part in the *American Journal of Mathematics* and in part in the *Bulletin des Sciences Mathematiques*. Let a system of $m+1$ points in $n+1$ dimensional space be given by the coördinates

$$x_{\nu,0}, x_{\nu,1}, \cdots, x_{\nu,n} \qquad (\nu = 0, 1, \cdots, m);$$

and let

$$x_{\nu,0} = x_{\nu,0}(x_{\nu,1}, \cdots, x_{\nu,n}), \quad x_{\nu,0,j} = \frac{\partial x_{\nu,0}}{\partial x_{\nu,j}}, \quad x_{\nu,0,j,k} = \frac{\partial^2 x_{\nu,0}}{\partial x_j \cdot \partial x_k} = x_{\nu,0,k,j}.$$

The general projective group of this $n+1$ dimensional space is generated by the $(n+1)(n+3)$ independent infinitesimal transformations

$$\frac{\partial f}{\partial x_i}, \quad x_i \frac{\partial f}{\partial x_k}, \quad x_i \Sigma\, x_j \frac{\partial f}{\partial x_j}, \qquad (i, j, k = 0, 1, \cdots, n).$$

By forming the second extensions of these point transformations by the method of Lie and equating them to zero, we have a complete system of partial differential equations whose integration will determine the projective differential invariants of the second order. The integration of this system for the case of $m+1$ points yields the result that the following $m+1$ forms are absolute invariants by the general projective group:

$$\sum_0^m {}^i \frac{|x_{k,0,1,1}, x_{k,0,2,2}, \cdots, x_{k,0,n,n}|}{|x_{i,0,1,1}, x_{i,0,2,2}, \cdots, x_{i,0,n,n}|} \left\{ \frac{x_{i,0} - x_{k,0} - \sum_1^n {}^j (x_{i,j} - x_{k,j}) x_{i,0,j}}{x_{i,0} - x_{k,0} - \sum_1^n {}^i (x_{i,j} - x_{k,j}) x_{k,0,j}} \right\}^{n+2},$$

$$(k = 0, 1, \cdots, m).$$

These invariants are interpreted geometrically in the following manner: Take $m+1$ hypersurfaces (n dimensional manifoldnesses) in the $n+1$ dimensional space, perfectly arbitrarily chosen except that a hypersurface is to pass through each of the $n+1$ points of the given system; let $\rho_{i,1}, \cdots, \rho_{i,n}$ be the principal radii of curvature of the hypersurface through the point $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ at this point; take any point $(x_{k,0}, x_{k,1}, \cdots, x_{k,n})$ of the system and join it by straight lines to all the other points of the system; let θ_i be the angle at $(x_{k,0}, \cdots, x_{k,n})$ between the normal to the hypersurface through this point and the line joining it to $(x_{i,0}, \cdots, x_{i,n})$, and let φ_i be the angle between the latter line and the normal to the surface through $(x_{i,0}, \cdots, x_{i,n})$ at the point; then the above invariants show that the forms

$$\sum {}^i \frac{\prod_{j=1}^{j=n} \rho_{k,j} \cos^{n+2} \theta_i}{\prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2} \varphi_i}$$

are absolute constants. When the $m+1$ points lie on a surface of the $(m+1)$th degree and on a straight line simultaneously

$$\sum_0^m {}^i \frac{1}{\prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2} \varphi_i} = 0.$$

In Professor Baker's first paper it is shown that, assuming the weight of a point as its defining property, there are only six possible operations performable on it. The processes of complex numbers and quaternions are not extensions but applications of these operations. In his second paper Professor Baker traces the fertility of many mathematical ideas to four fundamental characteristics of space.

The algebra of space considered by Dr. Hall is a vector algebra whose laws of operation are derived from geometric definitions of vector addition and vector multiplication.

Every real algebraic transformation corresponds to, and represents, a possible motion in space; and at any stage of the process algebraic expressions may be expressed geometrically, and vice versa. The algebra is developed far enough to enable it to be used in considering the properties of loci of one, two, and three dimensions.

Professor Moore's paper will be published in the *Mathematische Annalen*. The chief features of the paper are outlined in the following abstract. The modular group Γ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

of the complex variable ω, where the $\alpha, \beta, \gamma, \delta$ are rational integers, has for every rational prime q a self-conjugate subgroup $\Gamma_{\mu(q)}$ of finite index $\mu(q)$ containing all substitutions $(\alpha, \beta, \gamma, \delta)$ for which $\alpha \equiv 1, \beta \equiv 0, \gamma \equiv 0, \delta \equiv 1 \pmod{q}$. The corresponding quotient-group $\Gamma / \Gamma_{\mu(q)}$ is conveniently given as the say finite modular group $G^{q+1}_{\mu(q)}$ of substitutions $(\alpha, \beta, \gamma, \delta)$ on the $q+1$ marks $\omega(\omega = \infty, 0, 1, \cdots, q-1)$, where the $\alpha, \beta, \gamma, \delta$ are integers taken modulo q. By generalizing from the Galois field of rank 1 to that of rank n we have the generalized finite modular group $G^{q^n+1}_{M(q^n)}$ of order $M(q^n) = q^n(q^{2n}-1)$ or $q^n(q^{2n}-1)/2$ according as $q=2$ or $q>2$. Mathieu first exhibited this group and studied its cyclic subgroups (*Comptes Rendus*, 1858, 1859; *Liouville's Journal*, 1860, p. 39). That (except for the cases $q^n = 2^1, 3^1$) it is simple was proved by Moore (Mathematical Papers of the Chicago Congress, 1893; in abstract, *Bulletin of the New York Mathematical Society*, December, 1893), and independently, but imperfectly, by Burnside (*Proceedings of the London Mathematical Society*, February, 1894).

In the present paper all subgroups of the $G^{q^n+1}_{M(q^n)}$ are determined; for the case $n=1$ this was done by Gierster (*Mathematische Annalen*, vol. 18 (1881)). The subgroups are of three kinds: (1) metacyclic or solvable groups; (2) (3) groups of the abstract character of certain groups $G^{q^n+1}_{M(q^n)}$ or of certain groups $G^{q^n+1}_{2M(q^n)}$ $(q>2)$, a group $G^{q^n+1}_{2M(q^n)}$ being obtained by extending the $G^{q^n+1}_{M(q^n)}$ by the substitution $\omega' = \rho\omega$, where ρ is a primitive root of the $GF[q^n]$. Thus

the doubly infinite system of simple groups

$$G^{q^n+1}_{M(q^n)} \qquad (q^n+2^1, 3^1)$$

determines by the decomposition of the subgroups of its constituent groups, apart from the simple groups of prime order, only simple groups of the original system. An equation of degree $q^n+1 (q^n+2^1, 3^1)$ whose Galois group is the $G^{q^n+1}_{M(q^n)}$ has resolvents of degree D, $D < q^n+1$, only in the cases $q^n = 5^1, 7^1, 11^1, 3^2$, when D is respectively 5, 7, 11, 6. For $n=1$ this is a noted theorem of Galois and Gierster.

The group $\Gamma[\Omega]$ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$ where the coefficients are algebraic integers of an algebraic field Ω [Körper Ω] has for every prime functional π [Weber] of absolute norm q^n, a self-conjugate subgroup $\Gamma^n_{\pi(q)}[\Omega]$. The quotient-group is the group $G^{q^n+1}_{M(q^n)}$. The substitutions of the modular group Γ leave invariant a modular function $J(\omega)$. The transcendental modular equation $J = J(\omega)$ has algebraic resolvents, modular equations, with Galois groups $G^{q+1}_{M(q)=\mu(q)}$. The group $\Gamma[\Omega]$, for field Ω of degree > 1, is improperly discontinuous and has no corresponding automorphic function. If however an automorphic function $K(\omega)$ belongs to a group whose substitutions have coefficients of an algebraic field Ω, then the generalized modular equation $K = K(\omega)$ has algebraic resolvents whose Galois groups are certain generalized finite modular groups $G^{q^n+1}_{M(q^n)}$ or certain of their subgroups. This remark connects the present paper with recent work of Fricke and Bianchi on certain special classes of automorphic functions (Cf. Fricke-Klein, Automorphe Functionen I, p. 585 fg., 1897).

Professor Conant gave some illustrations of substitution properties by means of the shuffling of the cards of an ordinary whist pack.

Dr. Boyd's paper furnishes a method for finding an approximate general integral of the differential equations

$$(a) \qquad \frac{dy}{dx} + P_0 + P_1 y + P_2 y^2 = 0,$$

$$(b) \qquad P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0,$$

where the P's are any functions of x. The object of the paper is first to reduce the differential equations indicated under (a), (b), to the *normal Riccatian form*

$$\frac{dz}{dx} + z^3 = F(x),$$

then to develop a process, by the repetition of which an approximate integral is found. The paper assumes that an approximate integral is found when it is shown that the remainder, left in the equation after the so-called approximate integral has been substituted, becomes as small as we choose when the process has been repeated often enough.

Dr. Stecker's paper presented a discussion of the properties of non-euclidean cubics. Among the results obtained were the following: The ratio of the sines of the non-euclidean distances from any tangent of a cubic to either focal line and to the pole of that line with respect to the absolute equals $\sqrt{\mu}$. For any focal line and pole there are twelve tangents to the cubic such that the ratio of the sines of the non-euclidean distances from focal line and pole is constant. The ratio of the sines of the non-euclidean distances from any tangent of a cubic to any pair of focal lines equals $\sqrt{\frac{\mu}{\mu_1}}$ times the ratio of the sines of the non-euclidean distances to the poles of those lines with respect to the absolute. The non-euclidean distances from the point of contact of a cubic with any absolute tangent to the intersections with that tangent of any pair of focal lines are equal; the absolute tangent to the cubic makes equal angles with lines joining its point of contact to either pair of foci. The ratio of the product of the cosines of the non-euclidean distances from any point of a cubic to the poles, with respect to the absolute, of the tangents at three collinear points of inflexion to the cube of the cosine of the non-euclidean distance to the pole of the line through the three points of inflexion is constant. If two of the above tangents are absolute tangents, then the ratio of the square of the cosine of the non-euclidean distance from any point of a cubic to the intersection of such a pair of absolute tangents to the product of the cosines of the non-euclidean distances to their poles with respect to the absolute is constant. The relation between the cosines of the non-euclidean distances of any point of a cubic from the poles with respect to the absolute of any pair of focal lines (no two passing through the same absolute point) is expressed by the vanishing of the determinant

$$\begin{vmatrix} \lambda\Omega_{xo} & \cos xo \\ \lambda_1\Omega_{yo} & \cos yo \end{vmatrix}$$

for every point of the cubic. The product of $\lambda\Omega_{xo}$ into the cosine of the non-euclidean distance from any point of a cubic to the pole with respect to the absolute of any focal line is the same in value for all focal lines. If a is the intersection of two absolute tangents and x, y, z the poles, with respect to the absolute, respectively of the chord of contact, third tangent, and satellite line, then the ratio of any two products of the form $\cos^2 ao \cos^2 xo \cos yo \cos zo$ is constant for all points of the cubic for any pair of absolute tangents. The ratio of the product of the sines of the non-euclidean distances from any point of a cubic to the focal lines of any pair equals $\frac{\lambda}{\lambda_1}$ times the corresponding ratios for any other pair. The determinant

$$\begin{vmatrix} \sin\frac{1}{xo} \sin\frac{1}{yo} & \lambda \\ \sin\frac{1}{x_1o} \sin\frac{1}{y_1o} & \lambda_1 \end{vmatrix}$$

vanishes for any pair of focal lines of a cubic. Among other properties discussed was a class of relations concerning first and second polars with respect to the cubic of fixed lines related to the cubic and absolute.

By far the simplest definition of the first hypoabelian group consists in regarding it as the group of linear substitutions on m pairs of indices ξ_i, η_i taken modulo 2, which leave invariant the function

$$\xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

By Jordan's proof this group has a subgroup of index two which is *simple* if $m > 2$. The first generalization of this group, made by Dr. Dickson while at Paris and presented a year and a half ago to the *Quarterly Journal*, has since been greatly simplified and published in the July number of the BULLETIN. The present abstract of Dr. Dickson's first paper, to be presented for publication to the London Mathematical Society, announces a further generalization. Use is made of the notion of a Galois field, viz., the totality of polynomials built upon the root of a congruence of degree n and irreducible modulo p. These polynomials, of which p^n are

distinct, have the property that the addition, subtraction, multiplication, or division (except by zero) of two polynomials gives always an unique third polynomial, belonging to the set. The generalization may now be described as the replacement of the field of integers modulo 2 by the Galois field of order p^n, n any integer and p any prime. Consider the group $L_{m,n,p}$ of all linear substitutions

$$S\begin{cases}\xi_i' = \sum_{j=1}^{m}(\alpha_{ij}\xi_j + \gamma_{ij}\eta_j),\\ \eta_i' = \sum_{j=1}^{m}(\beta_{ij}\xi_j + \delta_{ij}\eta_j),\end{cases} \qquad (i = 1, \cdots, m).$$

(in which coefficients and indices belong to the Galois field of order p^n) which leave invariant

$$\xi_1\eta_1 + \cdots + \xi_m\eta_m$$

The quadratic conditions thus imposed on the coefficients are readily written down and the fact verified that the determinant of S must be ± 1. It is shown that the group $L_{m,n,p}$ may be generated from the substitutions

$$Q_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\eta_i,$$
$$T_{i,\lambda}: \quad \xi_i' = \lambda\xi_i, \quad \eta_i' = \lambda^{-1}\eta_i,$$
$$E_i = (\xi_i\eta_i);$$

that it has a subgroup $L'_{m,n,p}$ of index two and order

$$O'_{m,n,p} =$$
$$[p^{nm}-1][(p^{n(2m-2)}-1)p^{n(2m-2)}][(p^{n(2m-4)}-1)p^{n(2m-4)}]$$
$$\cdots[(p^{2n}-1)p^{2n}].$$

which is extended to the main group by one of the transpositions $E_j \equiv (\xi_i\eta_i)$. Thus, if $p > 2$, $L'_{m,n,p}$ contains all the substitutions of L having determinant $+1$; while for $p = 2$, it is readily proven to contain all satisfying the relation

$$\sum_{i,j}^{1,\cdots,m}\alpha_{ij}\delta_{ij} = m.$$

The group $L'_{m,n,p}$ contains the invariant subgroup (of index one or two)

$$L''_{m,n,p} \equiv \{Q_{i,j,\lambda},\ E_iE_j\} \quad (i, j = 1, \cdots, m;\ i \neq j).$$

Indeed, it is proved to contain the products $T_{i,\lambda}\ T_{j,\lambda}$ and consequently (if $m \geqq 3$) also

$$T_{1,\lambda^2} \equiv (T_{1,\lambda}\ T_{2,\lambda})\ (T_{2,\lambda^{-1}}\ T_{2,\lambda^{-1}})\ (T_{2,\lambda}\ T_{1,\lambda}).$$

Whether or not it contains the substitution $T_{i,\nu}$, ν being a not-square, is not, as yet, determined. It requires a lengthy, detailed investigation to prove that $L''_{m,n,p}$ contains no self-conjugate subgroup other than itself and the identity, unless it be the group of order 2 generated by the substitution changing the sign of every index, viz.,

$$T_{1,-1}\ T_{2,-1} \cdots T_{m,-1}.$$

The structure of the group $L_{m,n,p}$ defined by the invariant $\xi_1\eta_1 + \cdots + \xi_m\eta_m$ is therefore fully determined aside from the lack of proof of the non-existence in the group $\{Q_{i,j,\lambda}, E_iE_j\}$ of a substitution $T_{i,\nu}$, ν being a not-square. For the case $p = 2$, there is no uncertainty, since every quantity in the $GF[2^n]$ is a square. We thus know that $L''_{m,n,2} \equiv L'_{m,n,2}$ is a *simple* group of order

$$(2^{n_m}-1)\ [(2^{n(3m-2)}-1)2^{n(3m-3)}] \cdots [(2^{2n}-1)2^{2n}].$$

Judging from the proofs found for certain cases (when $p > 2$), it seems probable that the substitution $T_{i,\nu}$ does not occur in the group $L''_{m,n,p}$. If this be true, the orders of the simple quotient-groups are for $p > 2$ as follows:

$\frac{1}{2}O'_{m,n,p}$ if m be odd and -1 a not-square;

$\frac{1}{4}O'_{m,n,p}$ in all other cases.

Dr. Westlund's paper is in abstract as follows: Taking the equation for the division of the periods of the elliptic functions as starting point we are led to a class of equations of transformation whose roots are the $n + 1$ values of

$$y_{\frac{\mu}{\nu}} = \prod_{1,m}^{p} sn^{2\alpha} \cdot cn^{\beta} \cdot dn^{\gamma}(4p\bar{\omega}/k)$$

$$\bar{\omega} = \frac{\mu K + \nu iK'}{n}$$

$$\alpha, \beta, \gamma = \text{integers}, \quad n = \text{prime} = 2m + 1.$$

The object of the paper is to give a method by which all equations of this class may be computed. 1. The roots are

developed into q-series $\left(q = e^{\pi i\tau}, \tau = \frac{iK'}{K}\right)$ 2. A superior limit of the degree of x in the equation is determined by determining all the infinities (and their order) of the roots for the whole infinite x-plane ($x = k^2$). 3. Equations that can be derived from a given equation of transformation by means of linear transformations are considered. 4. The equations whose roots are

$$\prod_{1,n}^{p} \frac{dn^2}{cn}(4p\bar{\omega}/k), \prod_{1,n}^{p} \frac{cn}{sn^2}(4p\bar{\omega}/k), \prod_{1,n}^{p} sn^2 \cdot cn \cdot dn(4p\bar{\omega}/k),$$

respectively and all equations that can be derived from these by linear transformations are computed for $n = 3$.

To find the day of the week corresponding to a given date Dr. Stabler gives the following rule: add to the day of the month the index number of the month and the index number of the year, then subtract the largest multiple of seven that is less than the sum. Index numbers of months are: January, 3 (in leap years, 2); February, 6 (in leap years, 5); March, 6; April, 2; May, 4; June, 0; July, 2; August, 5; September, 1; October, 3; November, 6; December, 1. Index numbers of years 1800–1899 are found by increasing the excess of the year over 1800 by $\frac{1}{4}$ of itself (discarding fractions) and subtracting the largest multiple of seven contained in the sum. A rule is given for dates in other centuries. Example, August 19, 1898. Add index number of the month, 5, and of the year, 3, to 19. Excess over a multiple of 7 is 6, indicating Friday.

Dr. Martin's two papers will be published in the *Mathematical Magazine*. The author gives convenient processes for the extraction of roots and for determining the number corresponding to a given *natural* logarithm.

F. N. COLE.

COLUMBIA UNIVERSITY.

NOTE ON THE GENERALIZATION OF POINCARÉ AND GOURSAT'S PROOF OF A THEOREM OF WEIERSTRASS'S.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

WEIERSTRASS* enunciated the theorem that to an arbitrary (single-leaved) continuum A there correspond single-valued functions, analytic at every point of A, but not capable of being continued beyond the boundary of A. The first published proof of this theorem appears in a memoir by Runge,† in which the author considers the problem in its most general form and constructs a proof by methods so valuable in themselves that the paper is an important contribution to the general theory of functions. As early, however, as 1881 Poincaré‡ and almost simultaneously with him, but independently,§ Goursat‖ had devised a proof for the case that the continuum is bounded by a curve that has curvature everywhere,—at least, this requirement is made explicitly by Poincaré, and Goursat needs a part of it. The two authors thus assume the theorem with virtually the same restrictions, and their proofs, which are identical in substance, differ only slightly in form. At that time the subject of Cantor's sets of points was new to mathematicians,—in fact, this very problem and problems of a similar character treated by Mittag-Leffler¶ contributed largely toward making Cantor's theory known by showing some of its applications in the general theory of functions,—and so this proof of Poincaré and Goursat failed to receive the generalization of which its nature ren-

* *Berliner Monatsberichte*, Aug., 1880; Zur Functionenlehre, 1886, p. 92; Mathematische Werke, vol. 2, p. 223.

† "Zur Theorie der eindeutigen analytischen Functionen," *Acta Math.*, vol. 6 (1885).

‡ "Sur les fonctions à espaces lacunaires," *Acta soc. sci. Fennicæ*, vol. 12, Helsingfors; reprinted, under the same title, with additional matter, in the *Amer. Journ.*, vol. 14 (1892). Cf. also Hermite, Cours d'Analyse, 3d ed., p. 157, or 4th ed., p. 171.

§ Cf. second reference under ‖.

‖ *Comptes Rendus*, 13 March. 1882; the proof there given is reproduced in greater detail in a paper entitled, "Sur les fonctions à espaces lacunaires," *Bull. d. Sci. Math.*, vol. 22 (1887).

¶ "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendente;" *Acta Math.*, vol. 4 (1884).

ders it readily capable. The object of this note is to give this generalization.

It may be remarked that Mittag-Leffler had the necessary means at his disposal for the proof of Weierstrass's theorem in his memoir of 1884 above referred to. Let Q be an arbitrary, isolated set of points, infinite in number, and let Q' be the derivative of Q. Let Q' form the boundary of a continuum A, consisting of but one piece, within which the points Q lie. Mittag-Leffler showed that there exists a single-valued function, analytic within A, which vanishes at each point of Q, and nowhere else, and which therefore has in each point of Q' an essential singularity. In order to obtain Weierstrass's theorem from this result it is sufficient to show that, an arbitrary continuum A, consisting of a single piece, being given, it is always possible to choose an isolated set of points Q, lying within A, in such a manner that the derivative, Q', coincides with the boundary of A. This theorem had been proved by Bendixson,* and Mittag-Leffler, in considering a theorem relating to the approximate representation of a given function, refers to this paper.† But that which is essential in Weierstrass's theorem is that it is an *existence theorem*—a continuum A is arbitrarily given and there exist functions having A as their domain of definition. This point of view, for the most general case, is lacking in Mittag-Leffler's paper. Mittag-Leffler nowhere refers to Weierstrass's theorem in its general form.

Let C be any set of points whatever forming the complete boundary of a continuum A lying in the x-plane, and let $x = \infty$ be an interior point of A. A set of points $b_0, b_1, \cdots$, each belonging to C, shall now be chosen in such a manner that each point of C not itself a b-point shall have b-points clustering about it, and that furthermore, if these are taken as the b-points of Poincaré's formula (1), p. 203,‡ the proof there given that the function $\varphi(x)$ cannot be continued beyond A will hold with certain modifications to be pointed out.

Let the x-plane be divided up into squares, 2^{-s} on a side, by lines drawn parallel to the axes of reals and pure imaginaries, these axes themselves belonging to these lines. In a square containing points of A and, in its interior or on its boundary, a point of C, let a point x' of A be

*"Un théorème auxilaire de la théorie des ensembles;" *Bihang till Kongl. Svenska Vet. Ak. Handlingar*, vol. 9, No. 7 (1884).

†l. c., p. 45.

‡*Amer. Jour.*, l. c.

chosen in such a manner that its distance from some points of C is not greater than its distance from the boundary of the square. About x' as center draw a circle so small that it contains no points of C, and let its radius grow till one or more points of C lie on its circumference, but none inside it. Choose one of these points of C as a point b. It will lie within or on the boundary of the square, since the circle does not extend beyond the boundary of the square. The number of squares corresponding to a given value of n, in which such b-points lie, is finite.

Now begin with $n = 1$ and denote the corresponding b-points, taken in any order, by $b_0, b_1, \cdots, b_{r_1-1}$. Next, let $n = 2$. In each of those new squares that contain points of A and C, but no one of the points $b_0, \cdots b_{r_1-1}$, mark a new point b as above defined, and denote these new b-points, taken in any order, by $b_{r_1}, b_{r_1+1}, \cdots, b_{r_2-1}$. And so on indefinitely. Then every point of C not itself a b-point has b-points clustering about it.

Form Poincaré's function:

$$\sum_{n=0}^{\infty} \frac{A_n}{x - b_n} = \varphi(x),$$

where the series of constants $\sum_{n=0}^{\infty} A_n$ converges absolutely. Let x_0 be any point of A, R the radius of that circle about x_0 as center that contains at least one point of C on its circumference, but no point of C in its interior. Then Poincaré's analysis applies, without modification, to the proof that $\varphi(x)$ is analytic at least within this circle. The special case that $x_0 = \infty$ presents no difficulty.

Passing now to the proof that $\varphi(x)$ is not analytic throughout a larger circle about x_0, Poincaré's assumption that, to begin with, x_0 shall lie on a normal to C in a point b_k shall be modified as follows: To the point b_k as defined in the present note there corresponded a point x' of A such that the circle about x' passing through b_k contained no point of C in its interior, but possibly an infinite number of points of C on its circumference. As point x_0 we will therefore choose a point lying on the line joining x' with b_k and situated between these points. The circle described with this point x_0 as center and passing through b_k will contain no point of C in its interior and only one point, namely, b_k, on its circumference. To the power series

$$-\varphi(x) = \sum_{q=0}^{\infty} B_q (x - x_0)^q$$

representing $-\varphi(x)$ within this circle all of Poincaré's analysis applies without modification. Hence this circle is the true circle of convergence for this series.

Finally, for the case that x_0 is any point of A, Poincaré's reasoning, with the modification just given, still holds, and the theorem is thus established that $\varphi(x)$ is analytic in A, but cannot be continued beyond A.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

SUPPLEMENTARY NOTE ON A SINGLE-VALUED FUNCTION WITH A NATURAL BOUNDARY, WHOSE INVERSE IS ALSO SINGLE-VALUED.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

IN the June number of the BULLETIN I gave an example of a single-valued function with a natural boundary, the inverse of which is also single-valued. The function employed was the following:

$$f(z) = z + \frac{z^{a+2}}{(a+1)(a+2)} + \frac{z^{a^2+2}}{(a^2+1)(a^2+2)} + \cdots,$$

where a is a positive integer greater than unity. This function is continuous within and on the boundary of the unit circle, is analytic within this circle, and cannot be continued analytically beyond it.

I am indebted to Professor Hurwitz for an exceedingly simple proof of the principal theorem of my note, namely, that the inverse function is single-valued. The point to be established is that, z, z' being any two distinct points within or on the unit circle,

$$f(z) \neq f(z').$$

This follows at once by the application of a method employed by Professor Fredholm* to show that the inverse of the function

* *Cf.* Verhandlungen des ersten internationalen Mathematiker-Kongresses in Zürich vom 9. bis 11. August 1897; herausgegeben von Dr. Ferdinand Rudio, Professor am eidgenössischen Polytechnikum; Teubner, 1898; p. 109.

$$F(z) = \sum_{n=0}^{\infty} a^{n} z^{a^{n}}, \quad (|a| < 1),$$

is single-valued, provided $|a|$ is not too large.

The proof is as follows. Evidently

$$\left|\frac{f(z) - f(z')}{z - z'}\right| = \left|1 + \sum_{n=1}^{\infty} \frac{z^{a^{n}+1} + z^{a^{n}} z' + \cdots + z'^{a^{n}+1}}{(a^{n}+1)(a^{n}+2)}\right|$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{|z|^{a^{n}+1} + |z|^{a^{n}} |z'| + \cdots + |z'|^{a^{n}+1}}{(a^{n}+1)(a^{n}+2)}$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{1}{a^{n}+1} = 1 - \frac{1}{a+1} - \left(\frac{1}{a^{2}+1} + \frac{1}{a^{3}+1} + \cdots\right)$$

$$> 1 - \frac{1}{a+1} - \frac{1}{a(a-1)} > 0.$$

Hence $|f(z) - f(z')| > 0,$ q. e. d.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

NOTE ON THE PERIODIC DEVELOPMENTS OF THE EQUATION OF THE CENTER AND OF THE LOGARITHM OF THE RADIUS VECTOR.

BY PROFESSOR ALEXANDER S. CHESSIN.

IF we put with Professor S. Newcomb*

$$E = ev_1 + e^2 v_2 + e^3 v_3 + \cdots \tag{1}$$

$$\rho - \log a = e\rho_1 + e^2 \rho_2 + e^3 \rho_3 + \cdots \tag{2}$$

where E stands for the equation of the center and $\rho = \log r$, then v_i and ρ_i will be of the form

$$iv_i = \tfrac{1}{2}\Sigma k_j^{(i)} \sin j\zeta, \tag{3}$$

$$i\rho_i = \tfrac{1}{2}\Sigma h_j^{(i)} \cos j\zeta, \tag{4}$$

$$(j = i, \quad i-2, \quad i-4, \quad \cdots \quad , -i),$$

* "Development of the perturbative function," Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, vol. 5, part I., p. 12.

the coefficients $k_j^{(i)}$ and $h_j^{(i)}$ being rational numerical fractions subject to the conditions

$$k_j^{(i)} = -k_{-j}^{(i)}; \quad h_j^{(i)} = h_{-j}^{(i)}.$$

We propose to give in this note formulas by which these coefficients can be computed for any value of i and j.

If we put

$$E = \sum_{i=1}^{i=\infty} H_i \sin i\zeta, \tag{5}$$

$$\rho - \log a = \tfrac{1}{2}A_0 + \sum_{i=1}^{i=\infty} A_i \cos i\zeta, \tag{6}$$

then the comparison with formulas (1)-(4) gives

$$H_i = \sum_{m=0}^{m=\infty} \frac{k_i^{(i+2m)}}{i+2m} e^{i+2m}, \tag{7}$$

$$A_i = \sum_{m=0}^{m=\infty} \frac{h_i^{(i+2m)}}{i+2m} e^{i+2m}. \tag{8}$$

On the other hand it can be shown* that

$$H_i = \frac{2\sqrt{1-e^2}}{i} \sum_j \sum_q \frac{i^q}{q!} \left(\frac{e}{2}\right)^{j+q} N_{-i,j,q} \tag{9}$$

where j and q assume all integral positive values (zero included) such that

$$j + q = i, \quad i+2, \quad i+4, \cdots$$

If we develop $\sqrt{1-e^2}$ and put

$$H_i^{(2m)} = \sum_j \sum_q \frac{i^q}{q!} N_{-i,j,q} \qquad (-i+j+q=2m), \tag{10}$$

then formula (9) becomes

$$H_i = \frac{2}{i}\left(\frac{e}{2}\right)^i \sum_{m=0}^{m=\infty} \left(\frac{e}{2}\right)^{2m} \left[H_i^{(2m)} - 2H_i^{(2m-2)} - \cdots - \frac{1.3\cdots(2m-3)}{m!} 2^m H_i^0\right]. \tag{11}$$

Comparing this formula with (7) we conclude that

* Tisserand: Mécanique Céleste, vol. 1, p. 243.

$$(12)\quad k_i^{(i+2m)} = \left(\frac{i+2m}{i}\right)\cdot\frac{1}{2^{i+2m-1}}\left[H_i^{(2m)} - 2H_i^{(2m-2)}\right.$$

$$\left. - \frac{1}{2!}2^2 H_i^{(2m-4)} - \cdots - \frac{1.3\cdots(2m-3)}{m!}2^m H_i^0\right].$$

By this formula the computation of the coefficients $k_j^{(i)}$ is reduced to the computation of Cauchy's numbers for which the author has given a general formula.*

In order to obtain a similar expression for the coefficients $h_j^{(i)}$ we must first derive a development in powers of the eccentricity for the coefficients A_r. To this end we remark that

$$\frac{d\rho}{de} = \frac{d\log r}{de} = \frac{dr}{rde} = \frac{1}{e}\left(\frac{a}{r}\right) - \frac{1-e^2}{e}\left(\frac{a}{r}\right)^2$$

On the other hand we have†

$$\frac{a}{r} = 1 + 2\sum_{i=1}^{i=\infty} J_i(ie)\cos i\zeta$$

$$\left(\frac{a}{r}\right)^2 = \frac{1}{\sqrt{1-e^2}} + \sum_{i=1}^{i=\infty} G_i^{(2)}\cos i\zeta$$

where $J_i(ie)$ is a Bessel's function and

$$(13)\quad G_i^{(2)} = 2\sum_j\sum_q \frac{i^q}{q!}\left(\frac{e}{2}\right)^{j+q} N_{ij,q}.\quad (j+q = i, i+2, i+4, \cdots).$$

Hence we may write that

$$\frac{d\rho}{de} = \frac{1-\sqrt{1-e^2}}{e} + \frac{1}{e}\sum_{i=1}^{i=\infty}\left[2J_i(ie) - (1-e^2)G_i^{(2)}\right]\cos i\zeta.$$

Now, it follows from (6) and (8) that

$$e\frac{d\rho}{de} = \tfrac{1}{2}e\frac{dA_0}{de} + \sum_{i=1}^{i=\infty}\sum_{m=0}^{m=\infty} e^{i+2m}h_i^{(i+2m)}\cos i\zeta$$

which, compared with the preceding formula, shows that

$$2J_i(ie) - (1-e^2)G_i^{(2)} = \sum_{m=0}^{m=\infty} e^{i+2m}h_i^{(i+2m)}$$

and we only need to find the coefficient of e^{i+2m} in the left hand side to obtain the required expression for $h_i^{(i+2m)}$.

* *Annals of Mathematics*, vol. 10, p. 1.

† Tisserand, Mécanique Céleste, vol. 1, pp. 224 and 242.

From (9) and (13) follows that

$$(1-e^2)G_i^{(i)} = i\sqrt{1-e^2}\,H_i$$

$$= 2\left(\frac{e}{2}\right)^i (1-e^2)\sum_{m=0}^{m=\infty} H_i^{(2m)}\left(\frac{e}{2}\right)^{2m}$$

$$= 2\left(\frac{e}{2}\right)^i \sum_{m=0}^{m=\infty} [H_i^{(2m)} - 4H_i^{(2m-2)}]\left(\frac{e}{2}\right)^{2m}$$

so that the coefficient of e^{i+2m} in $(1-e^2)\,G_i^{(i)}$ is found to be

$$\frac{1}{2^{i+2m-1}}[H_i^{(2m)} - 4H_i^{(2m-2)}]$$

while the coefficient of the same power of e in $2J_i(ie)$ is

$$(-1)^m \frac{1}{2^{i+2m-1}} \cdot \frac{i^{i+2m}}{m\,!\,(i+m)\,!}.$$

Hence, we conclude that

$$(14)\quad h_i^{(i+2m)} = \frac{1}{2^{i+2m-1}}\left[4H_i^{(2m-2)} - H_i^{(2m)} + \frac{(-1)^m i^{i+2m}}{m\,!\,(i+m)\,!}\right]$$

which is the desired expression for the coefficients $h_j^{(i)}$.

To conclude we will express the coefficients $h_j^{(i)}$ by means of the $k_j^{(i)}$. To this end we multiply formula (7) by $\sqrt{1-e^2}$ and develop the right hand side in powers of e. Thus we obtain

$$\sqrt{1-e^2}H_i = \sum_{m=0}^{m=\infty} e^{i+2m}\left[\frac{k_i^{(i+2m)}}{i+2m} - \left(\frac{1}{2}\right)\frac{k_i^{(i+2m-2)}}{i+2m-2} - \cdots\right.$$

$$\left. - \frac{1.3\cdots(2m-3)}{m\,!}\left(\frac{1}{2}\right)^m \frac{k_i^i}{i}\right]$$

d, therefore,

$$h_i^{(i+2m)} = \frac{2(-1)^m\left(\frac{i}{2}\right)^{i+2m}}{m!(i+m)!} - \frac{ik_i^{(i+2m)}}{i+2m} + \frac{1}{2}\frac{ik_i^{(i+2m-2)}}{i+2m-2}$$

$$+ \frac{1}{2!}\left(\frac{1}{2}\right)^2 \frac{ik_i^{(i+2m-4)}}{i+2m-4} \cdots + \frac{1.3\cdots(2m-3)}{m!}\left(\frac{1}{2}\right)^m \frac{ik_i^i}{i}$$

which formula enables us to compute the values of the $h_j^{(i)}$ directly from the $k_j^{(i)}$.

NEW YORK,
July 4, 1898.

THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (THIRD PAPER.)

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

THE following pages form a continuation of two papers presented under the same title to the Society and printed on pp. 295–313 and 365–376 of the preceding volume of the BULLETIN. These papers will be referred to as *Th. of Osc. 1* and *2* respectively.

The object of the present paper is to extend the results so far established to some cases in which the coefficients of the differential equation in question are no longer continuous throughout the intervals with which we are concerned. Such extensions are made in §§ 2 and 3 of the present paper but for this purpose it is necessary in § 1 to establish some fundamental theorems concerning linear differential equations of the second order with discontinuous coefficients, results which are perhaps of some interest apart from the special applications here made of them. Before entering on these questions, however, it will be convenient to describe accurately the kind of discontinuities with which we shall deal, and to establish certain general theorems.

All the functions with which we shall have to deal in the present paper are, throughout the interval in which we consider them, single valued *real* functions of one or more *real* variables. Taking first the case of a function of a single variable $f(x)$, we shall consider only the case in which this function has in an interval $a \leqq x \leqq b$ a finite number of points of discontinuity.

The simplest discontinuities from some points of view are the so-called *finite discontinuities** of $f(x)$, *i. e.*, discontinuities $x = c$ for which a positive quantity M can be found such that in the neighborhood of c, $|f(x)| < M$. Going beyond these finite discontinuities we have discontinuities at

* Strictly speaking this includes the case in which the discontinuity is simply due to the fact that the function has not been defined at the point in question; for example e^{-1/x^2} at the origin. We will, however, here once for all make the convention that in such cases we will regard the function as being so defined at the point in question as to preserve the continuity if possible.

which the function becomes infinite for some or for all methods of approach. For our purpose we need merely to restrict these discontinuities by requiring that the functions do not become infinite more strongly than $(x-c)^{-i}$ where $0<i<1$. We may then describe the discontinuities with which we have to deal, including the finite discontinuities, as follows:

(A) *The functions $f(x)$ with which we shall have to deal are such that a function $\psi(x)$ exists single valued and continuous throughout the interval $a \leqq x \leqq b$ and such that at every point of this interval except $c_1, c_2, \cdots, c_n$*

$$\psi(x) = [(x-c_1)(x-c_2)\cdots(x-c_n)]^i f(x),$$

where $0<i<1$.

It should be noticed that for functions of this sort $\int f(x)dx$ extended over a part or the whole of the interval ab is absolutely convergent, *i. e.*, $\int |f(x)|\,dx$ is convergent.

We shall also have to deal with functions of two or more independent variables $x, \lambda, \mu, \cdots$ which we can describe as follows:

(B) *The functions $f(x;\lambda,\mu,\cdots)$ with which we shall have to deal are such that a function ψ can be found which is a single valued and continuous function of $(x,\lambda,\mu,\cdots)$ when $a \leqq x \leqq b$, $\lambda_2 \leqq \lambda \leqq \lambda_1$, $\mu_2 \leqq \mu \leqq \mu_1, \cdots$ and such that for all of these values except when $x = c_1, c_2, \cdots, c_n$*

$$\psi(x;\lambda,\mu,\cdots) = [(x-c_1)(x-c_2)\cdots(x-c_n)]^i f(x;\lambda,\mu,\cdots)$$

where i is a constant satisfying the inequality $0<i<1$.

We give now an important theorem concerning such functions:

(C) *If $f(x;\lambda,\mu,\cdots)$ is a function of the kind described in* (B) *and if x_0 is a constant satisfying the inequality $a \leqq x_0 \leqq b$, then*

$$F(x;\lambda,\mu,\cdots) = \int_{x_0}^{x} f(x;\lambda,\mu,\cdots)dx$$

is a continuous function of $(x,\lambda,\mu,\cdots)$ for all the values in question.

In order to prove this we have merely to show that as $\Delta x, \Delta\lambda, \Delta\mu, \cdots$ approach zero independently of each other the difference:

$$\Delta F = F(x+\Delta x;\ \lambda+\Delta\lambda,\ \mu+\Delta\mu,\cdots) - F(x;\lambda,\mu,\cdots)$$

approaches zero. Now we can write

$$\Delta F = \int_{x_0}^{x+\Delta x} f(x;\ \lambda+\Delta\lambda,\ \mu+\Delta\mu,\ \cdots)dx - \int_{x_0}^{x+\Delta x} f(x;\ \lambda,\ \mu,\ \cdots)dx$$
$$+ \int_{x_0}^{x+\Delta x} f(x;\ \lambda,\ \mu,\ \cdots)dx - \int_{x_0}^{x} f(x;\ \lambda,\ \mu,\ \cdots)dx.$$

We have, therefore, the inequality

$$|\Delta F| \leqq \int_{x_0}^{x+\Delta x} |f(x;\ \lambda+\Delta\lambda,\ \mu+\Delta\mu,\ \cdots) - f(x;\ \lambda,\ \mu,\ \cdots)| \cdot |dx|$$
$$+ \int_{x}^{x+\Delta x} |f(x;\ \lambda,\ \mu,\ \cdots)| \cdot |dx|.$$

The second of these integrals clearly approaches zero as Δx approaches zero. The first may be written, if we use the same notation as is used in (B), in the form

$$\int_{x_0}^{x+\Delta x} |(x-c_1)\cdots(x-c_n)|^{-t} |\Delta\psi| |dx|$$
$$\leqq \int_{a}^{b} |(x-c_1)\cdots(x-c_n)|^{-t} |\Delta\psi| dx,$$

where $\Delta\psi = \psi(x;\ \lambda+\Delta\lambda,\ \mu+\Delta\mu,\ \cdots) - \psi(x;\ \lambda,\ \mu,\ \cdots)$.

If now we denote the greatest value of $|\Delta\psi|$ in the interval $a \leqq x \leqq b$ by ϵ we see that the integral we are considering does not exceed

$$\epsilon \int_{a}^{b} |(x-c_1)\cdots(x-c_n)|^{-t} dx.$$

But owing to the uniform continuity of ψ, ϵ clearly approaches zero when $\Delta\lambda$, $\Delta\mu$, $\cdots$ all approach zero. Accordingly the whole integral, and therefore ΔF, approaches zero, as was to be proved.

(D) *If $f(x;\ \lambda,\ \mu,\ \cdots)$ is a function of the kind described in* (B) *and $\varphi(x,\ \lambda,\ \mu,\ \cdots)$ is throughout the region in question a single valued continuous function of $(x, \lambda, \mu, \cdots)$, then $f.\varphi$ is a function of the kind described in* (B).

The proof of this theorem follows immediately from (B).

(E) *If throughout the interval $a \leqq x \leqq b$, $\varphi(x)$ is single valued and continuous and $\varphi(c) = 0$ $(a \leqq c \leqq b)$, then if $k > 0$ and x_0 is any point of ab*

$$\lim_{x=c}\left[(x-c)^k \int_{x_0}^{x} \frac{\varphi(x)}{(x-c)^{k+1}} dx\right] = 0.$$

In order to prove this let us take a point x_1 so near to c that when $|x-c| \leqq |x_1-c|$, $|\varphi(x)| < k\varepsilon$. Now denoting the expression which stands above in square parentheses by $F(x)$ we have

$$F(x) = (x-c)^k \int_{x_0}^{x_1} \frac{\varphi(x)}{(x-c)^{k+1}} dx + (x-c)^k \int_{x_1}^{x} \frac{\varphi(x)}{(x-c)^{k+1}} dx$$

so that when $|x-c| < |x_1-c|$

$$|F(x)| < |x-c|^k \int_{x_0}^{x_1} \frac{|\varphi(x)|}{|x-c|^{k+1}} |dx|$$

$$+ k\varepsilon |x-c|^k \int_{x_1}^{x} \frac{|dx|}{|x-c|^{k+1}}$$

$$= |x-c|^k \int_{x_0}^{x} \frac{|\varphi(x)|}{|x-c|^{k+1}} |dx| + \varepsilon \left[1 - \left(\frac{x-c}{x_1-c}\right)^k\right].$$

The second of these two terms is evidently less than ε while the first can be made less than ε by taking $|x-c|$ sufficiently small. No matter how small ε may be chosen $|F(x)|$ can be made less than 2ε by taking $|x-c|$ small enough, and this proves the theorem.

§ 1. *On Linear Differential Equations of the Second Order with Discontinuous Coefficients.*

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0, \tag{1}$$

in which throughout the interval $a \leqq x \leqq b$ the coefficients p and q are functions of the kind described in (A). We can, of course, not expect to find solutions y which satisfy (1) at *all* points of ab. We shall therefore seek functions which, together with their first derivatives are single valued and continuous throughout ab and except at the points of discontinuity ($x = c_1, c_2, \cdots, c_n$) of p or q satisfy (1).

In order to simplify matters let us first assume that there is only *one* point ($x = c$) of discontinuity in ab. It is at once clear from (A) that we can find a positive quantity M such that at all points of the interval $a \leqq x \leqq b$ (except c)

$$|p| < M|x-c|^{-1}, \quad |q| < M|x-c|^{-1}.$$

We will now prove that *the method of successive approximations enables us to find a function y which, together with its first derivative, is single valued and continuous throughout the interval $a \leqq x \leqq b$, which satisfies* (1) *at every point of this interval except at c, and which at c has the arbitrarily prescribed value γ while its derivative has the arbitrarily prescribed value γ'.**

To prove this we will let

$$Y_0 = \gamma + \gamma'(x - c)$$

and compute the quantities $Y_1, Y_2, \cdots$ from the formulæ

$$(2) \qquad Y_n' = -\int_c^x [pY_{n-1}' + qY_{n-1}]dx, \qquad Y_n = \int_c^x Y_n' \, dx,$$

where accents denote differentiation. It should be noticed that the functions Y_n' and Y_n as thus defined are continuous throughout the interval ab. We wish to prove that the series

$$(3) \qquad Y_0 + Y_1 + Y_2 + \cdots$$

converges throughout ab and represents the desired solution of (1). In order to prove this let us consider by the side of the series (3) the series formed by differentiating it term by term

$$(4) \qquad Y_0' + Y_1' + Y_2' + \cdots.$$

We will begin by proving (3) and (4) uniformly convergent throughout ab. Let us write for brevity $|x - c| = t$. Let l be a positive quantity greater than 1 and such that at all points of ab, $t < l$; and finally let C be a positive quantity such that at all points of ab the inequalities $|Y_0| < C$, $|Y_0'| < C$ hold. We will first establish the inequalities

$$(5) \qquad |Y_n'| < \frac{C}{n!}\left(\frac{2Ml}{1-i}t^{1-i}\right)^n, \qquad |Y_n| < \frac{C}{n!}\left(\frac{2Ml}{1-i}t^{1-i}\right)^n.$$

These formulæ evidently hold when $n = 0$. They will therefore be established for all values of n if assuming them

* A precisely similar theorem holds for homogeneous linear differential equations of order higher than the second. The proof is essentially the same as that here given.

to hold for a given value of n we can show that they hold for the next larger value. Now we have

$$|Y'_{n+1}| \leqq M \int_0^t t^{-i} [|Y'_n| + Y_n|] \, dt$$

$$< \frac{2CM}{n!} \left(\frac{2Ml}{1-i}\right)^n \int_0^t t^{n-(n+1)i} \, dt$$

$$= \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{1}{l} t^{(n+1)(1-i)}$$

$$< \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i} t^{1-i}\right)^{n+1}$$

Furthermore

$$|Y_{n+1}| \leqq \int_0^t |Y'_{n+1}| \, dt < \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{1}{l} \int_0^t t^{(n+1)(1-i)} dt$$

$$= \frac{C}{(n+1)! \, [(n+1)(1-i)+1]} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{t}{l} t^{(n+1)(1-i)}$$

$$< \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i} t^{1-i}\right)^{n+1}$$

Thus formulæ (5) are established. From these formulæ we can deduce the following inequalities in which the variable t no longer enters

$$(6) \qquad |Y'_n| < \frac{C}{n!} \left(\frac{2Ml^{1-i}}{1-i}\right)^n,$$

$$|Y_n| < \frac{C}{n!} \left(\frac{2Ml^{1-i}}{1-i}\right)^n.$$

From (6) it appears that each term in (3) and also each term in (4) is less than the corresponding term in the exponential series

$$\sum_{n=0}^{n=\infty} \frac{C}{n!} \left(\frac{2Ml^{1-i}}{1-i}\right)^n.$$

Since this last series is a series with positive constant terms and is known to be convergent, it follows that the series (3) and (4) are both uniformly convergent throughout ab. Each series, therefore, represents a continuous function of x and (4) represents the derivative of (3). We will denote

the value of the series (3) by y. We see then that y is a function which, together with its first derivative, is single valued and continuous throughout ab. To prove that y satisfies (1) at every point of ab other than c let us differentiate (4) term by term

$$-(pY_0' + qY_0) - (pY_1' + qY_1) - (pY_2' + qY_2) - \cdots.$$

It is clear that this same series might have been obtained by multiplying (4) by $-p$ and (3) by $-q$ and adding. It is accordingly uniformly convergent throughout any part of ab which does not contain c, and is therefore equal to y''. On the other hand it is equal to $-py' - qy$. Therefore at every point of ab other than c, y satisfies (1).

Finally when $x = c$ all the terms after the first in both (3) and (4) are zero while the first reduce to γ and γ' respectively. Thus our theorem is established.

We will now prove that *every function y which satisfies* (1) *at all points of the interval $a \leqq x < c$ (or of the interval $c < x \leqq b$) approaches, as does also its derivative y', a finite limit as x approaches c.* To prove this let y_1 and y_2 be two of the solutions just obtained by the method of successive approximations, chosen (and this is evidently possible) so as to be linearly independent of each other. Then $y = C_1y_1 + C_2y_2$ where C_1 and C_2 are two constants. Now since y_1, y_1', y_2, y_2' approach finite limits as x approaches c it follows that the same is true of y and y'.

Finally, we will prove that *if throughout the interval $a \leqq x \leqq b$, y_1 and y_2 are together with their first derivatives single valued and continuous and except when $x = c$ satisfy* (1), *and if $y_1(c) = y_2(c) = \gamma$ and $y_1'(c) = y_2'(c) = \gamma'$, then $y_1 = y_2$ throughout the interval ab.* For consider first the case in which γ and γ' are not both zero. Then if y_1 and y_2 were not identical it is clear that they must be linearly independent. Every other solution y would therefore be expressible in the form: $y = C_1y_1 + C_2y_2$, so that

$$y(c) = (C_1 + C_2)\gamma$$

and

$$y'(c) = (C_1 + C_2)\gamma'.$$

We thus see that $y(c)$ and $y'(c)$ are proportional to γ and γ' whereas we have proved that they can be arbitrarily chosen. This disposes of all cases except that in which $\gamma = \gamma' = 0$, and here the theorem above stated will be established if we can prove that y_1 and y_2 are both identically zero. This, however, follows at once from the case already

considered as otherwise by adding y_1 or y_2 to a solution y_3 which does not vanish (or whose derivative does not vanish) at c we should get a different solution which at c has the same value and the same derivative as y_3, and this we have just proved to be impossible.

The results just obtained can be at once extended to the case in which there is more than one point of discontinuity of p or q in the interval ab. For this purpose it is merely necessary to break up the interval ab into shorter intervals in each of which only one point of discontinuity lies. We thus get the following theorem

I. *If in the interval $a \leqq x \leqq b$, p and q are functions of the kind described in* (A) *there exists one and only one function y which together with its first derivative is single valued and continuous throughout ab, satisfies* (1) *at every point of ab except at the points of discontinuity of p or q, and has at an arbitrarily given point of ab (which may or may not coincide with one of the discontinuities of p or q) an arbitrarily given value while its derivative also has at this point an arbitrarily given value.*

Finally we note that theorems (E) and (F) *Th. of Osc. 1*, pp. 297, 298 hold for the more general case with which we are now dealing in which p and q are functions of the kind described in (A). The same is true of the theorem concerning the separation of the roots of two linearly independent solutions quoted in *Th. of Osc. 2*, p. 366.

In §3 we shall have to deal with the case in which in (1) the coefficient p is still a function of the kind described in (A)* while q has a point of discontinuity c at which it becomes infinite more strongly than the functions described in (A) but only in such a way that the function $(x-c)q$ belongs to the class described in (A). We will here again begin by confining ourselves to the case in which c is the only point of discontinuity of p or q in the interval $a \leqq x \leqq b$, and we will prove that in this case *the method of successive approximations enables us to find a function y_1 which together with its first derivative is single valued and continuous throughout the interval $a \leqq x \leqq b$, which satisfied (1) at every point of this interval, except at c, and which vanishes at c while its derivative has at c the arbitrarily prescribed value γ'.*†

* In fact in the special case with which we there have to deal $p = 0$.

†This is merely a special case of the following theorem which can be proved by the same method: If $p_1, p_2, \cdots, p_s, (x-c)p_{s+1}, (x-c)^2 p_{s+2}, \cdots, (x-c)^{n-1}p_n$ are functions of the kind described in (A) whose only discontinuity is at c, then a function Y exists and can be found by the method above used which with its first $n-1$ derivatives is single valued

In order to prove this we will take as a first approximation

$$Y_0 = \gamma'(x - c),$$

and form the functions $Y_1, Y_2, \cdots$ by means of the formulæ (2), p. 26. The first of the two integrals in (2) is now not obviously convergent since q has at c a discontinuity up to which we cannot in general integrate. In order to prove this integral convergent it is clearly sufficient to prove that

$$Y_n = (x - c)\,\varphi_n(x)$$

where $\varphi_n(x)$ is continuous at c. This formula holds when $n = 0$, for then we have $\varphi_0 = \gamma'$. If then, assuming that it is true for a given value of n, we can prove it true for the next larger value the convergence of our integral will be established. We will then assume the formula true when we give n the value $n - 1$. Then Y_n' is a continuous function of x when $a \leqq x \leqq b$, and the same is therefore true of Y_n. Remembering that $Y_n(c) = 0$ we have by the law of the mean

$$Y_n(x) = (x - c)\; Y_n'(\bar{x})$$

where $|\bar{x} - c| < |x - c|$. We have therefore $\varphi_n(x) = Y_n'(\bar{x})$, and since Y_n' is continuous at c and $Y_n'(c) = 0$ we see that $\varphi_n(x)$ approaches the limit zero as x approaches c. If then we define $\varphi_n(x)$ to have the value zero when $x = c$ we see that φ_n is continuous at c as well as elsewhere.

Having thus justified the definition of the quantities Y_n the next step is to prove that the series (3) and (4), p. 26 are uniformly convergent. For this purpose we need a few inequalities.

In the first place it is clear that we can find a positive quantity M such that at all points of ab (other than c)

$$|p| < M|x - c|^{-i}, \quad |x - c||q| < M|x - c|^{-i}.$$

We then establish by mathematical induction (the reasoning being almost identical with that used in the analogous case on p. 27) the following inequalities

---

and continuous throughout ab, which at every point of ab except c satisfies the differential equation:

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

and which at c vanishes together with its first $n - k - 1$ derivatives while the derivatives $\frac{d^{n-k} y}{dx^{n-k}}, \cdots \frac{d^{n-1} y}{dx^{n-1}}$ have arbitrarily prescribed values at c.

$$|Y'_n| < \frac{|\gamma'|}{n!}\left(\frac{2Mt^{1-i}}{1-i}\right)^n,$$

$$|Y_n| < \frac{|\gamma'|}{n!}\left(\frac{2Mt^{1-i}}{1-i}\right)^n t,$$

where as before we have let $|x-c| = t$. By means of these inequalities the uniform convergence of (3) and (4) is established at once as on p. 27, and as there it is shown that (3) therefore represents a function y_1 of the kind desired. Our theorem is thus proved.

A few simple facts about these functions y_1 must be noted before we go further. In the first place it is clear that if $\gamma' = 0$ y_1 is identically zero. Apart from this trivial case it is clear that y_1 *cannot have an infinite number of roots in the interval ab*, for these roots could (by (E) *Th. of Osc. 1*, p. 297) have no other point than c as a limiting point. and this would clearly require the vanishing of $y_1'(c)$.

In the second place *we can write*

$$y_1(x) = (x-c)E(x),$$

where $E(x)$ is single valued and continuous throughout ab and $E(c) = \gamma'$. For $E(x)$ will of course be continuous everywhere in *ab* except at c. In order to prove our theorem it is therefore sufficient to show that

$$\lim_{x=c} E(x) = \gamma'.$$

This follows at once from the law of the mean which tells us that $y_1(x) = (x-c)\,y_1'(\bar{x})$ where $|\bar{x}-c| < |x-c|$. We have then $E(x) = y_1'(\bar{x})$, so that

$$\lim_{x=c} E(x) = \lim_{\bar{x}=c} y_1'(\bar{x}) = \gamma'.$$

It is clear that the solutions y_1 so far obtained are linearly dependent, differing only in the value of the constant factor γ'. We proceed to prove that *any solution of* (1) *linearly independent of the solutions y_1 just obtained approaches a finite limit different from zero as x approaches c.*

In order to prove this let us remember that if x_0 is a point so near to c that $y_1 \neq 0$ when $|x-c| \leqq |x_0 - c|$

$$y_2 = y_1 \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x} p\,dx}}{y_1^2}\,dx$$

is a solution of (1) linearly independent of y_1. Writing as above $y_1 = (x-c)E(x)$ we have

$$y_2 = (x-c)E(x)\int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x} p\,dx}}{(x-c)^2[E(x)]^2}dx.$$

Now we will write

$$\frac{1}{[E(x)]^2}e^{-\int_{x_0}^{x} p\,dx} = K + F(x),$$

where K is a constant different from zero and $F(x)$ is single valued and continuous while $F(c) = 0$. We then get

$$y_2 = -KE(x) + KE(x)\frac{x-c}{x_0-c} + (x-c)E(x)\int_{x_0}^{x} \frac{F(x)}{(x-c)^2}dx.$$

As x approaches c it is clear that the first of these three terms approaches a finite limit different from zero, while the second approaches zero. By (E) the third also approaches zero. y_2 therefore approaches a limit different from zero. Now every solution of (1) linearly independent of y_1 may be written in the form: $y = C_1y_1 + C_2y_2$ where $C_2 \neq 0$ so that the truth of our theorem is evident.*

The functions y, with which we have been dealing may of course be extended continuously throughout larger intervals where p and q have other discontinuities than c provided that these discontinuities are of the kind described in (A). We are thus led to the following theorem:

II. *If in the interval $a \leqq x \leqq b$, p and $(x-c)q$ (where $a \leqq c \leqq b$) are functions of the kind described in* (A), *there exists one and except for a multiplicative constant only one, function y_1 which together with its first derivative is single valued and continuous throughout ab, satisfies the differential equation* (1) *at every point of ab except at the points of discontinuity of p and q, and vanishes when $x = c$.*

We have so far regarded the coefficients p and q as functions of x only. We shall, however, in what follows be obliged to regard them as involving also a parameter λ.

Let us first suppose that when $a \leqq x \leqq b$, $\lambda_1 \leqq \lambda \leqq \lambda_1$, p and q are functions of the kind deseribed in (B). We begin as before with the case in which there is only one value $x = c$ for which p or q is discontinuous, and apply the method of

* It may be added that the derivatives of solutions linearly independent of y_1 will not in general approach finite limits at c.

successive approximations regarding γ and γ' as constants.* In the analysis of pp. 26–28 we have merely to consider the nature of the dependence upon λ of the quantities which appear. Thus M may be taken as independent of λ. Y_n and Y_n' will be by (C) continuous functions of (x,λ). The analysis moreover proves that (3) and (4) are uniformly convergent not merely with regard to x but with regard to (x,λ) so that y and y' are continuous functions of (x,λ). This result we state as follows:

III. *If when* $a \leqq x \leqq b$ *and* $\lambda_2 \leqq \lambda \leqq \lambda_1$ *p and q are functions of the kind described in* (B), *the solution of* (1) *which together with its first derivative is constant for an arbitrarily given value* $x = x_0$ $(a \leqq x_0 \leqq b)$, *is together with its first derivative a continuous function of* (x,λ) *when* $a \leqq x \leqq b$, $\lambda_2 \leqq \lambda \leqq \lambda_1$.

Finally we come to the case of differential equations in which p and $(x-c)q$ are functions of the kind considered in (B), concerning which the following theorem may be stated:

IV. *If when* $a \leqq x \leqq b$ *and* $\lambda_2 \leqq \lambda \leqq \lambda_1$, *$p$ and $(x-c)q$ (where* $a \leqq c \leqq b$*) are functions of the kind described in* (B), *the solution of* (1) *which vanishes when* $x = c$ *and whose derivative has at this point a constant value is when* $a \leqq x \leqq b$ *and* $\lambda_2 \leqq \lambda \leqq \lambda_1$ *together with its first derivative a continuous function of* (x,λ).

The proof here is almost precisely the same as for theorem III. The only point which presents any difficulty is the proof that Y_n' is a continuous function of (x,λ). In order to prove this let us write as on p. 30

$$Y_n(x,\lambda) = (x-c)\varphi_n(x,\lambda)$$

so that

$$Y_n'(x,\lambda) = -\int_c^x [pY'_{n-1} + (x-c)q\varphi_{n-1}]\,dx.$$

If then we can prove that the functions φ_n are continuous functions of (x,λ) when $x = c$ (they are obviously continuous for other values of x) we could as before infer that the functions Y_n' are continuous functions of (x,λ). We shall of course here again use the method of mathematical induction: noting that $\varphi_0 = \gamma'$ is continuous, then assuming that $\varphi_1, \varphi_2, \cdots, \varphi_{n-1}$ are continuous. From this it follows that Y_n' is a continuous function of (x,λ). Now we have (see p. 30)

$$\varphi_n(x,\lambda) = Y_n'(\bar{x},\lambda)$$

where $|x-c| < |x-c|$, whence the continuity of φ_n when $x = c$ follows at once.

* The result and the method of proof would remain the same if γ and γ' were any continuous functions of λ.

§ 2. *Extension of Results of Previous Papers to Equations with Discontinuous Coefficients for which the Boundary Conditions may Remain General.*

Leaving for the next section equations for which we should have to base our results on Theorems II and IV of the last section, we consider in the present section extensions of our previous results based on Theorems I and III. Here we may state at once the theorem

V. *All the theorems stated in Th. of Osc. 1, §§ 1, 2, and Th. of Osc. 2, § 1, remain true for differential equations whose coefficients instead of being continuous functions of x or (x, λ) are functions of the kind described in* (A) *or* (B).*

The truth of this statement is at once obvious when we refer to the proofs of the theorems in question. These proofs would require only a few modifications which, in view of the results of the last section, are so obvious that it does not seem necessary to dwell upon the matter further.

There remain then to be generalized the results contained in *Th. of Osc. 2*, § 2. We start here from the differential equation (1) in which

$$q(x) = \chi(x) + \psi(x)[C_k x^k + C_{k-1}x^{k-1} + \cdots + C_0],$$

and where p, χ, and ψ are independent of the parameters $C_0, \cdots, C_k$. We have to consider the $k+1$ finite intervals $a_0b_0, a_1b_1, \cdots, a_kb_k$ such that

$$a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k,$$

and the $k+1$ points $\xi_0, \xi_1, \cdots, \xi_k$ such that $a_i < \xi_i < b_i$. We will now assume *first* that each of the functions

$$t_i = g_i \int_{\xi_i}^{x} e^{-\int_{\xi_i}^{x} p dx} dx + f_i \qquad (i = 0, 1, \cdots, k)$$

(where g_i is any constant not zero and f_i any constant) is a continuous function of x throughout the interval $a_i \leqq x \leqq b_i$. This will of course be the case if p is a function of the kind described in (A), but it will also be true in some other cases which we shall find important.† This first assump-

* In fact, in *Th. of Osc. 1*, Theorem VIII, and *Th. of Osc. 2*, Theorem III, it is not necessary that $\phi(x, \lambda)$ should be a function of the kind described in (B) for the whole interval $L > \lambda > l$. It is sufficient that this should be the case for every interval $\lambda_2 \leqq \lambda \leqq \lambda_1$ contained in the interval just mentioned.

†Cf. for instance p. 37.

tion makes the interval $a_i b_i$ of the x-axis correspond in a one to one manner to a finite interval $\sigma_i \tau_i$ of the t_i axis.

Secondly we will assume that

$$\chi e^{\int_{p dx}^{x}} \text{ and } \psi e^{\int_{p dx}^{x}} \qquad (i = 0, 1, \cdots, k)$$

are throughout the interval $\sigma_i \tau_i$ (including the ends) functions of t_i of the kind described in (A); and

Thirdly that $\psi(x)$ does not change sign in any one of these intervals or have an infinite number of roots in them.

Under these circumstances we have the following theorem

VI. *There is one and only one real determination of the parameters $C_0, C_1, \cdots, C_k$ for which the equation (1) has $k+1$ solutions $y_0, y_1, \cdots, y_k$ such that y_i $(i = 0, 1, \cdots, k)$ has just m_i roots in the interval $a_i < x < b_i$ and satisfies the conditions*

$$a_i' y_i(a_i) - a_i \left[\frac{dy_i}{dt_i}\right]_{x=a_i} = 0,$$

$$\beta_i' y_i(b_i) - \beta_i \left[\frac{dy_i}{dt_i}\right]_{x=b_i} = 0.$$

Here the quantities m_i are any integers positive or zero, and a_i, a_i', β_i, β_i' are any real quantities subject only to the restriction that a_i and a_i' are not both zero and that β_i and β_i' are not both zero.

The proof of this theorem is practically identical with the proof of Theorem IV., *Th. of Osc. 2** so that we will omit it here. It should be noticed that this theorem includes as special cases Theorems IV and V of *Th. of Osc. 2*.

An important application of Theorem VI is to the general linear differential equation of the second order which is everywhere regular

$$\frac{d^2y}{dx^2} + \left(\frac{1 - x_1' - x_1''}{x - e_1} + \cdots + \frac{1 - x_n' - x_n''}{x - e_n}\right)\frac{dy}{dx}$$

$$+ \frac{1}{f(x)}\left[\frac{x_1' x_1'' f'(e_1)}{x - e_1} + \cdots + \frac{x_n' x_n'' f'(e_n)}{x - e_n}\right. \tag{7}$$

$$\left. + C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \cdots + C_0\right] y = 0,$$

where $f(x) = (x - e_1)(x - e_2) \cdots (x - e_n)$.

* The greatest difference comes in in (a) *Th. of Osc. 2*, p. 373, where we must now prove ϕ_k to be not a continuous function of (t_k, C_k) but a function of the kind described in (B).

At this point we will define a term which we shall find it convenient to make use of in what follows. *By the fundamental solutions of* (7) *corresponding to the singular point* e_i *we shall understand the solutions of* (7) *which can be developed about* e_i *in the form**

$$y^{(1)} = (x-e_i)^{\kappa_i'}[1 + g_1'(x-e_i) + g_2'(x-e_i)^2 + \cdots],$$

$$y^{(2)} = (x-e_i)^{\kappa_i''}[1 + g_1''(x-e_i) + g_2''(x-e_i)^2 + \cdots].$$

If we have occasion to distinguish between these two solutions we shall do so by speaking of the first as corresponding to the exponent κ_i' the second to κ_i''. Moreover if we recall that a non-singular point c of (7) may be regarded as having exponents 1 and 0 we see that it will be natural to *understand by the fundamental solutions of* (7) *corresponding to a non-singular point* c *the solutions which can be developed about* c *in the forms*

$$y^{(1)} = (x-c) + g_2'(x-c)^2 + g_3'(x-c)^3 + \cdots,$$

$$y^{(2)} = 1 \qquad + g_2''(x-c)^2 + g_3''(x-c)^3 + \cdots.$$

We assume as before that the coefficients of (7) are real for real values of x (Cf. *Th. of Osc. 2*, p. 375 $(a) - (d)$); and letting $k \leqq n-2$ we consider the $k+1$ intervals $a_i \leqq x \leqq b_i (i = 0, 1, \cdots, k)$ where

$$a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k.$$

If no singular point lies in any of these intervals it is clear as stated in *Th. of Osc. 2*, p. 375, that the simple form of Klein's theorem of oscillation is at once applicable. Let us now enquire how it is if one of the intervals $a_i b_i$ just reaches up with one of its ends to one of the real singular points of (7). Let us say for the sake of distinctness $a_i = e_i$. There are two cases to consider:

1. The case in which the exponents κ_1', κ_1'' are conjugate imaginaries. Here it is at once obvious from the form of development about e_1 that every real solution of (7) has an infinite number of roots in the neighborhood of e_1 so that the theorem of oscillation cannot be extended to this case. We need then merely to consider:

2. The case in which κ_1', κ_1'' are real. We will at first consider here only the special case (to which, however, as

* In case $\kappa_i' - \kappa_i''$ is an integer, that one of these series into which the smaller of the exponents enters would have to be replaced by a more complicated expression which, however, we shall not have occasion to use.

we shall see presently the general case can be reduced*) in which $\varkappa_1'' = 0$ while $\varkappa_1' > 0$. Here

$$t_i = g_i \int_{\xi_i}^{x} e^{-\int_{\xi_i}^{x} \left[\frac{1-\varkappa_1'}{x-e_1} + \frac{1-\varkappa_2'-\varkappa_2''}{x-e_2} + \cdots + \frac{1-\varkappa_n'-\varkappa_n''}{x-e_n}\right] dx} dx + f_i$$

$$= (x-e_1)^{\varkappa_1'} E(x) + \sigma_i,$$

where $E(x)$ is continuous throughout the interval $e_1 \leqq x \leqq b_i$ and $E(e_1) \neq 0$. The condition of continuity of t_i is therefore satisfied since by hypothesis $\varkappa_1' > 0$.

A relation which we shall need in a moment can at once be deduced from the value of t_i just obtained, viz:

$$x - e_1 = (t_i - \sigma_i)^{\frac{1}{\varkappa_1'}} E_1(t_i)$$

where E_1 is a single valued and continuous function of t_i throughout the interval $\sigma_i \tau_i$ including the ends, and

$$E_1(\sigma_i) \neq 0.$$

Going a step further we have

$$\chi(x) = \frac{1}{f(x)}\left[\frac{\varkappa_2'\varkappa_2'' f'(e_2)}{x-e_2} + \cdots + \frac{\varkappa_n'\varkappa_n'' f'(e_n)}{x-e_n} + C_{n-2}x^{n-2} + \cdots + C_{k+1}x^{k+1}\right],$$

$$\psi(x) = \frac{1}{f(x)},$$

where $f(x) = (x-e_1)(x-e_2)\cdots(x-e_n)$, and finally

$$e^{2\int_{\xi_i}^{x} p\,dx} = k(x-e_1)^{2-2\varkappa_1'}(x-e_2)^{2-2\varkappa_2'-2\varkappa_2''}\cdots(x-e_n)^{2-2\varkappa_n'-2\varkappa_n''}$$

where k is a constant different from zero. We may therefore write, using the value of $x - e_1$ in terms of t_i above obtained

$$\chi(x)e^{2\int_{\rho}^{x} dx} = (x-e_1)^{1-2\varkappa_1'} F_1(x) = (t_i-\sigma_i)^{\frac{1}{\varkappa_1'}-2}\Phi_1(t_i),$$

$$\psi(x)e^{2\int_{\rho}^{x} dx} = (x-e_1)^{1-2\varkappa_1'} F_2(x) = (t_i-\sigma_i)^{\frac{1}{\varkappa_1'}-2}\Phi_2(t_i),$$

*Except the case of equal exponents which as we shall see is excluded.

where F_1 and F_2 are both continuous throughout the interval $e_1 \leqq x \leqq b_i$, and Φ_1 and Φ_2 are both continuous throughout the interval $\sigma_i \tau_i$ including the ends. The functions

$$\chi(x) \cdot e^{\int_{\xi_i}^{x} \frac{y}{p} dx} \quad \text{and} \quad \psi(x) \cdot e^{\int_{\xi}^{x} \frac{y}{p} dx}$$

are therefore functions of t_i of the kind described in (A) provided that $0 < \varkappa_1' < 1$. *If this last inequality holds theorem VI may therefore be applied.* Moreover it is evident that instead of allowing one of the points $a_0, b_0, a_1, b_1, \cdots$ to coincide with a singular point of (7) we might have allowed any number of them to do so provided that one of the exponents of each of the singular points thus involved is zero while the other is real, greater than zero, and less than one.

The conditions to which the solutions y_i are subjected at the points $a_i b_i$ may be stated in the cases with which we are now concerned in a very simple form if we use the conception of the fundamental solutions corresponding to these points. Suppose for instance that the singular point e_1 is one of the points a_i, then y_i satisfies the condition

$$a_i' y_i(e_1) - a_i \left(\frac{dy_i}{dt_i} \right)_{x = e_1} = 0.$$

Now y_i can be written $k^{(1)} y^{(1)} + k^{(2)} y^{(2)}$ where $k^{(1)}$ and $k^{(2)}$ are constants and $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions corresponding to the exponents $\varkappa_1'$ and 0 respectively at e_1. Substituting this value of y_i into the above condition we readily see, when we use the explicit expression for t_i in terms of x, that the condition reduces to

$$a_i' k^{(2)} - a_i K k^{(1)} = 0$$

where K is a constant different from zero and independent of the parameters C. Now since a_i' and a_i can be chosen as any two constants not both zero we see at once that *the condition can be expressed by saying that y_i is to be proportional to an arbitrarily chosen linear combination of the fundamental solutions corresponding to e_1.*

The condition $\gamma' y(c) - \gamma y'(c) = 0$ at a non-singular point may also be stated in this form, for letting $y = k^{(1)} y^{(1)} + k^{(2)} y^{(2)}$, where $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions corresponding to c, we see at once that our condition reduces to $\gamma' k^{(2)} - \gamma k^{(1)} = 0$, and this is equivalent to requiring that y be proportional to an arbitrarily chosen linear combination of $y^{(1)}$ and $y^{(2)}$.

So far we have merely considered the case in which one of the exponents of each of the singular points with which we have to deal is zero. Let us now consider the more general case in which the exponents $\varkappa_i'$ and $\varkappa_i''$ of these points are any real quantities where $\varkappa_i' > \varkappa_i''$. We will let $\varkappa_i' - \varkappa_i'' = \varkappa_i$; and denote by $\Pi(x-e_i)^{\varkappa_i''}$ the product of the expressions $(x-e_i)^{\varkappa_i''}$ taken for all the singular points which coincide with any of the points a_i or b_i. Now let

$$y = \Pi(x-e_i)^{\varkappa_i''} \cdot \bar{y}$$

and we get as the differential equation satisfied by y an equation differing from (7) in only two respects:

1st. the exponents of the points e_i which coincide with points a_i or b_i are now $\varkappa_i$ and 0 instead of $\varkappa_i'$ and $\varkappa_i''$.

2d, the parameters $C_0, C_1, \cdots C_{n-2}$ of the transformed equation are connected with the parameters $C_0, C_1, \cdots, C_{n-2}$ of the original equation by relations of the form: $\bar{C}_i = C_i + k_i$ where the constants k_i are independent of the C's.

We may therefore determine the parameter $\bar{C}$ by applying the theorem of oscillation to the transformed equation provided that $0 < \varkappa_i < 1$; and the $\bar{C}$'s being thus determined the C's are uniquely determined. It remains then merely to show that we can so arrange the conditions of the theorem of oscillation when we apply it to the transformed equation as to give any desired conditions for the original equation.

In the first place it is clear that the roots of a solution y_i in an interval $a_i < x < b_i$ are precisely equal to those of the corresponding solution $\bar{y}_i$ in this interval, and therefore in particular that y_i and $\bar{y}_i$ have the same number of roots in this interval. Moreover if $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions of (7) corresponding to a point c (whether this point be singular or non-singular is immaterial) $y^{(1)}$ and $y^{(2)}$ although not themselves the fundamental solutions of the transformed equation differ from them only by the same constant factor. We may then state the following theorem:

VII. *If the coefficients of equation* (7) *are real for real values of* x *and if the* $k+1$ *intervals* $(k \leqq n-2)$ $a_0b_0, a_1b_1, \cdots, a_kb_k$ *are so situated that*

1st, $a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k$;

2d, *no singular point of* (7) *lies in any of the intervals*

$$a_i < x < b_i (i = 0, 1, \cdots, k);$$

3d, *the points a_i and b_i themselves, so far as they are singular points of* (7), *have unequal real exponents whose difference is numerically less than* 1;

Then, the quantities m_i being any integers positive or zero, there is one and only one real determination of the parameters $C_0, C_1, \cdots, C_k$ for which (7) *has $k+1$ solutions $y_0, y_1, \cdots, y_k$ such that y_i has just m_i roots in the interval $a_i < x < b_i$ and at each extremity of this interval is proportional to an arbitrarily chosen linear combination of the fundamental solutions corresponding to this extremity.*

§ 3. *Extension of Results of Previous Papers to Equations with Discontinuous Coefficients for which the Boundary Conditions must be Specialized.*

We will begin here by considering equations of the form:

$$\frac{d^2y}{dx^2} = \varphi \cdot y \tag{8}$$

where the coefficient φ is such that throughout the interval $a \leqq x \leqq b$, $(x-a)\varphi$ is a function of the kind described in (A) or (B). Here we see by Theorems II and IV that we can no longer require of solutions y that they shall satisfy the conditions $y(a) = \alpha$, $y'(a) = \alpha'$ where α and α' are arbitrarily given, but that we must now restrict ourselves to the boundary condition $y(a) = 0$. With this restriction, however, many of the results of the previous papers may be extended to the case now before us by methods practically identical with those there given. We may say:

VIII. *All the theorems contained in Th. of Osc. 1, § 1, 2 as well as Theorem III, Th. of Osc. 2 remain true if instead of φ being a continuous function of x or (x, λ), $(x-a)\varphi$ is a function of the kind described in* (A) *or* (B), *provided we let $a = 0$ in these theorems.*

Theorems almost identical with these would of course hold if the points a and b were interchanged (Cf. *Th. of Osc. 1*, § 2, last paragraph).

The methods used in the previous papers are not, however, applicable to the case in which φ becomes infinite both at a and at b more strongly than the functions described in (A) or (B). We shall, therefore, need a somewhat new method to prove the following theorem of oscillation:

IX. *If when $a \leqq \ldots \leqq b$ and $L > \lambda > l$,* $(x-a)(x-b)\varphi(x, \lambda)$ is a function of the kind described in* (B), *and if when $a \leqq x \leqq b$*

* We may have $L = +\infty$ or $l = -\infty$ or both L and l may be infinite. Cf. also the first footnote on p. 34 which applies here.

and $L > \lambda' > \lambda'' > l$, $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$;† *if, moreover*

$$\lim_{\lambda = L} \varphi(x, \lambda) = + \infty \text{ and } \lim_{\lambda = l} \varphi(x, \lambda) = - \infty$$

except perhaps for a finite number of values of x *in the interval* $a \leqq x \leqq b$ *; then there will be one and only one value of* λ *in the interval* $L > \lambda > l$ *for which* (8) *has a solution which has just* m *roots* (m *being any integer positive or zero*) *in the interval* $a < x < b$ *and which vanishes when* $x = a$ *and when* $x = b$.

In order to prove this let us take a point c so that $a < c < b$. Let us denote by y_1 and y_2 solutions of (8) which vanish at a and b respectively. It will be convenient to speak in the following proof of the *aggregate number of roots of* y_1 *and* y_2 *in* ab meaning thereby the integer obtained by adding to the number of roots of y_1 in the interval $a < x < c$ the number of roots of y_2 in the interval $c < x < b$ and increasing this number by 1 if y_1 or y_2 (or both) vanish at c. Our theorem will clearly be proved if we can show that there is one and only one value of λ in the interval $L > \lambda > l$ for which y_1 and y_2 are linearly dependent while this aggregate is m. For if y_1 and y_2 are linearly dependent they can be made equal by multiplying one of them by a suitable constant.

Now we see from VIII (and *Th. of Osc.* 2, theorem III) that there is one and only one value of λ in the interval $L > \lambda > l$, say λ_{m_1}', for which $y_1(x, \lambda_{m_1}')$ has m_1 roots in the interval $a < x < c$ while $y_1(c, \lambda_{m_1}') = 0$. There is also one and only one value of λ in the interval $L > \lambda > l$, say λ_{m_2}'', for which $y_2(x, \lambda_{m_2}'')$ has m_2 roots in the interval $c < x < b$ while $y_2(c, \lambda_{m_2}'') = 0$. Let us consider the two sequences $\lambda_0', \lambda_1', \lambda_2', \cdots$, and $\lambda_0'', \lambda_1'', \lambda_2'', \cdots$. Each of these sequences is arranged in order of decreasing magnitude. Let us now arrange the terms of the two sequences in a single sequence $\lambda_0, \lambda_1, \lambda_2, \cdots$, in order of decreasing magnitude; it being understood that if $\lambda_i' = \lambda_j''$ these two terms shall be represented by only one term in the new sequence. Let us denote by μ_k the aggregate number of roots of $y_1(x, \lambda_k)$ and $y_2(x, \lambda_k)$ in ab. Then it is clear that in general $\mu_{k+1} = \mu_k + 1$, the only exception being that if $y_1(c, \lambda_k) = y_2(c, \lambda_k) = 0$, $\mu_{k+1} = \mu_k + 2$. In the sequence of integers $\mu_0, \mu_1, \cdots$, one at least of the two integers m and $m - 1$ must, therefore, be present. Let us

† This inequality need not of course hold for the points of discontinuity of φ. Moreover, there must be some points (besides these poinst of discontinuity) where the equality sign does not hold.

give to k the value which makes $\mu_k = m$ if this is possible, and if not the value which makes $\mu_k = m - 1$.

One case we can dispose of at once, viz., that in which $\mu_k = m$ and $y_1(c, \lambda_k) = y_2(c, \lambda_k) = 0$. When $\lambda > \lambda_k$ the aggregate number of roots of y_1 and y_2 in ab is less than m, when $\lambda < \lambda_k$ it is greater than m. Accordingly when $\lambda = \lambda_k$ and for this value only is the aggregate equal to m. Moreover, for this value y_1 and y_2 are linearly dependent since they both vanish when $x = c$. In this case then our theorem is proved.

Apart from this case, which we now exclude, it is clear that $\lambda = \lambda_k$ is not the value of λ we are seeking. Leaving aside this value then we see that the aggregate number of roots of y_1 and y_2 in ab is equal to m when and only when $\lambda_k > \lambda > \lambda_{k+1}$. It remains then to prove that in this range of values of λ there is one and only one value for which y_1 and y_2 are linearly dependent, *i. e.* (since $y_1(c, \lambda) \neq 0$ and $y_2(c, \lambda) \neq 0$) for which $y_1'(c, \lambda)/y_1(c, \lambda) = y_2'(c, \lambda)/y_2(c, \lambda)$. This follows at once from *Th. of Osc. 1*, theorem II for as λ decreases from λ_k to λ_{k+1} $y_1'(c, \lambda)/y_1(c, \lambda)$ will be decreasing continuously while $y_2'(c, \lambda)/y_2(c, \lambda)$ is increasing continuously. Moreover we must have either $y_1(c, \lambda_k) = 0$ or $y_2(c, \lambda_k) = 0$ and either $y_1(c, \lambda_{k+1}) = 0$ or $y_2(c, \lambda_{k+1}) = 0$. Accordingly when $\lambda = \lambda_k$ and also when $\lambda = \lambda_{k+1}$ one of the two ratios y'/y just considered is infinite. There are four possibilities here, viz.: 1st, y_1'/y_1 decreases from $+\infty$ to $-\infty$; 2d, y_1'/y_1 decreases from $+\infty$ to a finite limit while y_2'/y_2 increases from a finite limit to $+\infty$; 3d y_1'/y_1, decreases from a finite limit to $-\infty$ while y_2'/y_2 increases from $-\infty$ to a finite limit; 4th, y_2'/y_2 increases from $-\infty$ to $+\infty$. In any case, however, it is clear that the two ratios become equal once and only once, as was to be proved.

In this proof we have tacitly assumed that $m \neq 0$. The changes necessary in the case $m = 0$ are however so slight and so obvious that it does not seem necessary to dwell upon them.*

The generalization of Klein's theorem of oscillation follows now so readily that we merely give the result:

X. *Theorem* VI *still holds if we drop at some or all of the points* a_i *and* $b_i (i = 0, 1, \cdots, k)$ *the requirements that*

$$\chi \cdot e^{\int_{t_i}^{x} p dx} \quad \text{and} \quad \psi \cdot e^{\int_{t_i}^{x} p dx}$$

* Were it not for this case $m = 0$ we might have followed a slightly simpler method analogous to that used by Picard in a somewhat similar case. Cf. Traité d'Analyse, vol. 3, p. 122.

be functions of the kind described in (A) *and merely require that these functions when multiplied by* $x - a_i$ (*or by* $x - b_i$) *are functions of the kind described in* (A) PROVIDED, *that at the points* a_i *and* b_i *just mentioned we require that* $\alpha_i = 0$ *and* $\beta_i = 0$.

If we apply this theorem to equation (7) we get

XI. *Theorem* VII *still holds if we allow some or all of the points* a_i *and* b_i *to be singular points with unequal real exponents whose difference is no longer restricted to being numerically less than* 1, PROVIDED *that at each of these points we require that the corresponding solution* y_i *should be proportional to the fundamental solution corresponding to the greater exponent of this point.*

Theorems VII and XI include all the cases for which Klein has discussed the theorem of oscillation in his lithographed lectures: Lineare Differentialgleichungen der zweiten Ordnung, 1894, or for which he has there expressed the opinion, based on analogy with simpler cases, that the theorem is probably true.

We close by noting two slight generalizations:

1st The proof of theorems VII and XI would not be in the least affected if instead of the polynomial

$$C_{n-2}x^{n-2} + \cdots + C_0$$

which enters equation (7) we had a polynomial of degree greater than $n-2$, in which case the point $x = \infty$ would be an irregular point. The number $k+1$ of intervals $a_i b_i$ might of course then be correspondingly increased.

2d Theorems VII and XI still remain true, the proofs not being essentially affected, if several of the points $e_1, e_2, \cdots, e_n$ coincide. provided that none of the intervals $a_i b_i$ reaches up to such a "multiple" point, which will obviously in general be an irregular point.

[*Note added Sept. 14, 1898:* It was not until the present paper was in print that I noticed that some of the theorems contained in it had been proved for a particular differential equation and in somewhat restricted form in a paper by Van Vleck "On the Polynomials of Stieltjes" in the June number of the BULLETIN. There is even an analogy of method between the proof of Van Vleck's Theorem IV and that of my Theorem IX. It may therefore be well to state that I was in possession of all the results and methods of the present paper last February (Cf. a remark in the middle of p. 365 *Th. of Osc. 2*) and was prevented from working them out in detail only by press of other work.]

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

NOTES.

THE American Association for the Advancement of Science will hold its annual meeting in 1899 at Columbus, Ohio. DR. ALEXANDER MACFARLANE is vice-president, and JOHN F. HAYFORD secretary, for the section of mathematics and astronomy. Professor EDWARD ORTON, of Ohio State University, was elected president of the Association.

THE British Association for the Advancement of Science will hold its meeting for 1899 at Dover, on September 13–20. The French Association will meet at about the same time on the opposite side of the channel, at Boulogne, so that active intercourse between the two associations is likely to be an attractive feature of the meetings.

INTERNATIONAL CONGRESS OF MATHEMATICIANS AT PARIS IN 1900. The Mathematical Society of France has elected a committee of organization and decided to hold the congress from the 6th to the 12th of August, 1900. The committee of organization is made up of two committees. one occupied with the scientific works and the other with the administrative functions. Of the former committee, M. POINCARÉ is president, MM. APPELL and PICARD, vice-presidents, and M. RAFFY, secretary; of the latter committee, M. DARBOUX is president, MM. HATON DE LA GOUPILLIÈRE and VICAIRE, vice-presidents, MM. DUPORCQ and LAISANT, secretaries, and M. DÉSIRÉ ANDRÉ, treasurer. The president of the society, M. LECORNU, is also at the head of the joint-committee. The headquarters of the commission are those of the Mathematical Society, 7, rue des Grands-Augustins, Paris, to which address all communications relative to the international congress of mathematicians should be sent.

THE French translation of PROFESSOR WEBER'S Algebra, undertaken by M. J. GRIESS, has just been published by Gauthier-Villars, of Paris. PROFESSOR E. PASCAL, of the University of Pavia, has issued the first volume of his "Repertorio di matematiche superiori" through the press of U. Hoepli, of Milan; the work is to be complete in two volumes, occupied with analysis and geometry respectively.

A FACSIMILE of the Rhind mathematical papyrus, is to be issued by the trustees of the British museum.

AT the last meeting of the French Association for the Advancement of Science at Nantes in August of this year

about thirty thousand francs were voted as grants in aid of scientific work, thirteen thousand francs being from the funds of the Association, and nineteen thousand from the Girard legacy fund.

An anonymous donor has placed at the disposal of the Council of the University of Paris a sum of fifteen thousand dollars, to be renewed yearly during the next three or four years, for the founding of five travelling scholarships of three thousand dollars each. These scholarships are to be held by graduates contemplating secondary teaching as a career, and one of the chief conditions attached is that the holders of the scholarships must be able to speak English.

The Berlin Academy of Sciences, at its meeting this year in celebration of the birthday of Leibnitz, announced the following prize problem for 1902 :

"Let $f_1(z), f_2(z), \cdots f_n(z)$ be a fundamental system of integrals of a linear homogeneous differential equation with algebraic coefficients. A thorough investigation is demanded of the function of the variables $\frac{u_2}{u_1}, \frac{u_3}{u_1}, \cdots, \frac{u_n}{u_1}$, which is defined by means of the equation

$$u_1 f_1(z) + u_2 f_2(z) + \cdots + u_n f_n(z) = 0.$$

In particular a representation of the function is to be ascertained for the case in which z is a finite-valued function. The discussion is to be included of the question to what extent these particular functions can be made of value in the integration of linear differential equations of the nth order."

Also the following prize problem, on the Eller legacy foundation, was set for 1904 :

"A new method for the determination of the solar constant is to be devised, or one of the known methods so far improved, that the influence of the variable distance between the earth and the sun in observations made at different times of the year be unequivocally recognizable. The method chosen is to be verified by series of observations sufficiently extensive, embracing at least three perihelia and three aphelia."

These problems were previously proposed in different forms, the first at the Leibnitz meeting of 1894, the second at that of 1892. Both remained unsolved. Solutions will be received in German, Latin, French, English, or Italian up to the thirty-first of December, 1901, and 1903, respectively. No manuscript revealing the name of the author will be accepted. Each manuscript is to bear a mark or *nom-de-plume*,

and to be accompanied by a sealed envelope containing the name and address of the author and bearing outside the corresponding mark or assumed name. The prize for the solution of the first problem is five thousand marks, that of the second, two thousand marks. Manuscripts should be sent to the Bureau of the Academy, Berlin NW. 7, Universitätsstrasse, 8.

THE royal mathematical prize, given by the King of Italy, has been recently awarded to Professors CORRADE SEGRE and VITO VOLTERRA. The prize was divided equally between them. There were eight competitors, who submitted about ninety written and printed memoirs. The award of the astronomical prize has been deferred for a period of two years, but a sum of three thousand lire has been awarded to Professor FILIPPO ANGELITTE in consideration of his editing and discussing the unpublished writings of Professor Carlo Brioschi.

UNIVERSITY OF BERLIN. The following courses in mathematics are announced for the winter semester of 1898–99 :—By Professor FUCHS: Theory of elliptic functions, four hours; Theory of linear differential equations, four hours; Seminar, two hours.—By Professor SCHWARZ: Differential calculus, four hours; Exercises, two hours; Calculus of variations, four hours; Special problems, two hours; Colloquium, two hours; Seminar, two hours.—By Professor FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours.—By Professor KNOBLAUCH: Theory of curved surfaces, four hours; Theory and application of determinants, four hours; Exercises, one hour.—By Professor HENSEL: Theory of numbers, four hours; Integral calculus, four hours; Theory of surfaces and curves in space, two hours.—By Professor HOPPE: Analytic geometry, four hours; Differential calculus, four hours.

UNIVERSITY OF GÖTTINGEN. The announcements for the coming winter semester embrace the following courses in mathematics :—By Professor KLEIN: Theory of functions, four hours; Seminar, in conjunction with Professor Hilbert, two hours.—By Professor HILBERT: Mechanics, four hours; Theory of determinants, two hours; Elements of Euclidean geometry, two hours; Theory of functions of real variables, together with Professor Klein, two hours.—By Professor SCHOENFLIES: Projective geometry, four hours; Exercises in descriptive geometry, two hours; Proseminar, one hour.—By Dr. BOHLMANN: Integral calculus, four hours;

Mathematical principles of insurance, three hours; Exercises and seminar in the latter, two hours.

UNIVERSITY OF VIENNA. The courses in mathematics offered for the winter semester are as follows:—By Professor v. ESCHERICH: Theory of functions, three hours; Differential equations, three hours; Proseminar, one hour; Seminar, one hour.—By Professor GEGENBAUER: Elements of the infinitesimal calculus, with special reference to the needs of chemists, physicists and natural scientists, five hours; Exercises on the latter, one hour; Proseminar, one hour; Seminar, two hours.—By Professor MERTENS: Theory of numbers, five hours; Proseminar, one hour; Seminar, two hours.—By Professor KOHN: Introduction to synthetic geometry, four hours; Algebraic curves, one hour.—By Dr. SERSAWY: Lectures on the mathematics of insurance, three hours.—By Dr. TAUBER: Spherical harmonics and their application to mathematical physics, three hours; Mathematics of insurance, four hours; Exercises on the latter, two hours.—By Dr. ZINDLER: Elements of kinematics, two hours.—By Dr. BLASCHKE: Introduction to mathematical statistics, three hours.—By Dr. ZSIGMONDY: Application of the infinitesimal analysis to geometry, two hours; Fourier's series, one hour.—By Dr. DAUBLEBSKY v. STERNECK: Application of the calculus to geometry, two hours.

PARIS FACULTY OF SCIENCES. The mathematical courses offered during the first semester of the current academic year are the following:—Professor G. DARBOUX: Theory of triply orthogonal systems.—Professor E. GOURSAT: Differential and integral calculus and its applications to infinitesimal geometry.—Professor P. APPELL: General laws of equilibrium and motion.—Professor V. J. BOUSSINESQ: Theory of elasticity.—Professor G. KOENIGS: Kinematics of solid or deformable bodies.—Mr. RAFFY: Mathematics introductory to various scientific disciplines.—Mr. ANDOYER: Theory of the determination of the orbits of planets and comets, and theory of special perturbations. In addition, Professor H. POINCARÉ conducts conferences in the new theories of electrodynamics, and particularly in the theory of Lorentz. Other mathematical conferences are conducted by Messrs. HADAMARD, PUISEUX, ANDOYER and BLUTEL.

For the second semester the following courses have been announced:—Professor PICARD: Partial differential equations from the point of view of mathematical physics.—Professor GOURSAT: Differential equations.—Professor POINCARÉ: Figures of the heavenly bodies and their motions

about their centers of gravity.—Professor BOUSSINESQ: Equilibrium of elasticity of the sphere; propagation of motion in an elastic and homogeneous medium of infinite dimensions.—Professor KOENIGS: Study of machines. Professor APPELL will also lecture, but the subject of his course has not been announced.

IN resigning the professorship of geometry in the University of Leipzig to accept a professorship of mathematics in the University of Christiania, PROFESSOR SOPHUS LIE returns to his alma mater. At the age of sixteen he entered the University of Christiania in 1859; passed his "Statsexamen" in December, 1865; received his doctor's degree in July, 1871, the subject of his dissertation being "Om en classe geometriske transformationer." In the latter year he became "Universitetsstipendiat i mathematik" (privat docent), which post he resigned in 1886 to take the chair offered by the University of Leipzig.

PROFESSOR BARTHOLOMEW PRICE, Master of Pembroke College, Oxford, has resigned the Sadlerian professorship of natural philosophy, after a tenure of forty-five years.

PROFESSOR J. R. EASTMAN, who has been continuously connected with the United States Naval Observatory since 1862, has retired.

THE deaths of PROFESSOR PAUL SERRET, of Paris, and of DR. CHARLES E. EMERY, non-resident professor of engineering in Cornell University, are announced.

PROFESSOR D. HILBERT, of the University of Göttingen, has declined a call to the University of Leipzig. A similar call has been extended to PROFESSOR O. HÖLDER, of the University of Königsberg.

PROFESSOR K. ZORAWSKI has been promoted to a full professorship of mathematics at the University of Krakau.

AMONG the foreign members recently elected to the Reale Academia dei Lincei are PROFESSORS A. G. GREENHILL and V. VOIGT, in mechanics, and PROFESSOR W. C. RÖNTGEN, in physics.

THE senior wrangler at Cambridge University this last year was MR. R. H. W. T. HUDSON, of St. John's College, son of W. H. H. Hudson, professor of mathematics in King's College, London. MISS CAVE-BROWN-CAVE, of Girton, was bracketed fifth wrangler.

ASSISTANT PROFESSOR BARTLETT has been promoted to an associate professorship of mathematics in the Massachusetts Institute of Technology, and DR. A. COHEN, to the grade of associate in mathematics in Johns Hopkins University.

MR. J. N. FELLOWS has been appointed to the head professorship of mathematics in the Missouri State University to succeed Professor W. C. Tindall who resigned on account of failing health.

DR. J. M. PAGE has been promoted from an adjunct to an associate professorship of mathematics at the University of Virginia.

PROFESSOR ARNOLD EMCH, formerly of the Polytechnic School at Biel, Switzerland, has been appointed professor of graphic mathematics at the Kansas State Agricultural College.

MR. L. C. WALKER has been elected to the professorship of mathematics in the Montana State Normal School.

Among recent academic appointments are the following: MR. F. ATHELING, assistant in mathematics, Leland Stanford University; MR. B. S. EATON, instructor in mathematics, University of Iowa; DR. J. G. HARDY, instructor in mathematics, Williams College; MR. J. B. PROCTOR, assistant in mathematics, Darmouth College; DR. E. W. RETTGER instructor in mathematics, University of Indiana; DR. WILCZNSKI, instructor in mathematics, University of California.

DR. J. S. AMES has been promoted to a full professorship in physics in Johns Hopkins University. MR. ERNEST RUTHERFORD, of Trinity College, Cambridge, has been elected professor of physics in McGill University, Montreal. PROFESSOR E. F. NICHOLS, of Colgate University, has accepted a call to the chair of physics at Dartmouth College. DR. C. E. MENDENHALL (Johns Hopkins, '98) has been made instructor in physics at Williams College.

DR. H. A. SAYRE, of Ursinus College, has been elected professor of physics and astronomy in the State University of Alabama. DR. S. D. TOWNLEY, formerly instructor in astronomy in the University of Michigan, has been made instructor in astronomy at the University of California.

PROFESSOR HAMLIN, after twenty-six years' service as a member of the faculty of the State University of Maine and

seventeen years at the head of its department of civil engineering, has resigned. MR. N. C. GROVER is his successor. The chair of electrical engineering in McGill University has been filled by the election of PROFESSOR R. B. OWENS, of Nebraska State University.

THE University of Vienna has established a chair of mathematical statistics and insurance.

AT the seventeenth congress of German naturalists and physicians, which opened at Düsseldorf, September 19, Professor KLEIN gave an address on universities and technical high schools.

THE last number, volume 12, number 2, of Enestrom's *Bibliotheca Mathematica* contains an eight-page contribution, by G. VALENTIN, to the bibliography of the writings of Euler, composed of additions to Hagen's Index operum Leonardi Euleri, Berlin, Dames, 1896.

THE tenth Congress of Russian Naturalists and Physicians opened at Kieff on September 3d, with an attendance of nearly fifteen hundred members, under the presidency of PROFESSOR N. A. BUNGE. A paper on the philosophical purports of mathematics was read by PROFESSOR BUGAEFF at the opening meeting of the section of mathematics. The president of this section is PROFESSOR V. P. ERMAKOFF.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARZELÁ (C.). Sulla rappresentazione approssimata delle funzioni analitiche; nota. Bologna, Gamberini e Parmeggiani, 1898. 8vo. 11 pp.

BACHMANN (P.). Zahlentheorie. Versuch einer Gesammtdarstellung dieser Wissenschaft in ihren Haupttheilen. Theil IV: Die Arithmetik der quadratischen Formen, Abtheilung I. Leipzig, Teubner, 1898. 8vo. 16 and 668 pp. M. 18.00

BENDIXSON (J.). Sur les points singuliers des equations différentielles. 3 parties. Stockholm, 1898. 8vo. 17, 13 and 18 pp.

BESANT (W. H.). Elementary conics. London, Bell, 1898. 8vo. 184 pp. 2s. 6d.

BOREL (E.). Leçons sur la théorie des fonctions. Paris, Gauthier-Villars, 1898. 8vo. 8 and 138 pp. Fr. 3.50

BOURGUET (H.). Sur une classe supérieure de groupes hyperabéliens. Paris, 1898. 4to. 95 pp.

BUDISAVLJEVIC (E.) und MIKUTA (A.). Leitfaden für den Unterricht in der höheren Mathematik. Band I. Grundzüge der Determinanten-Theorie und der projectivischen Geometrie; analytische Geometrie. Wien, Braumüller, 1898. 8vo. 10 and 492 pp. M. 8.00

DALWIGK (F. v.). Ueber die Integration von $\Delta u = 0$, und die konforme Abbildung. Marburg, 1898. 8vo. 53 pp.

DRACH (J.). Essai sur une théorie générale de l'intégration et sur la classification des transcendantes (thèse). Paris, Gauthier-Villars, 1898. 4to. 150 pp.

DÜRLL (W.). Die Probleme des logarithmischen Potentials für eine von zwei Kreisbogen begrenzte ebene Fläche. [Dissertation.] Leipzig, 1898. 8vo. 134 pp.

ERLER (W.). Die Elemente der Kegelschnitte in synthetischer Behandlung. 5. Auflage von L. Heubner. Leipzig, 1898. 8vo. 6 and 60 pp. M. 1.20

FIEDLER (W.). See SALMON (G.).

FRICKE (F.). Ueber ebene Curven dritter Ordnung, welche durch die imaginären Kreispunkte gehen. [Dissertation.] Jena, 1898. 8vo. 33 pp.

GRIESS (J.). See WEBER (H.).

GRAF (J. H.) und GUBLER (E.). Einleitung in die Theorie der Bessel'schen Funktionen, in zwei Heften. 1. Heft: Die Bessel'schen Funktionen erster Art. Bern, Wyss, 1898. 8vo. 4 and 142 pp. M. 3.20

GRÖTSCH (C.). Störungstheorie und Berührungstransformationen. [Dissertation.] Leipzig, 1898. 8vo. 48 pp.

GROSS (T.). See STURM (C.).

GUBLER (E.). See GRAF (J. H.).

HALL (W. S.). Elements of the differential and integral calculus. New York, Van Nostrand Co., 1898. 8vo. 260 pp. Cloth. $2.25

HARKNESS (J.) and MORLEY (F.). Introduction to the theory of analytic functions. New York, The Macmillan Co., 1898. 8vo. 15 and 336 pp. Cloth. $3.00

HERMITE (C.). See RIEMANN (B.).

HOLMGREN (E.). Om Differentialekvationen $\Sigma_{ik}(x, y) \frac{d^{i+k}z}{dx^i dy^k} = 0$. Upsala, 1898. 8vo. 33 pp. M. 1.20

KLEIN (F.). See RIEMANN (B.).

KOBER (G.). Die Grundgebilde der neueren Geometrie. Eine geordnete Zusammenstellung ihrer Um- und Abbildungen erster und zweiter Ordnung (2 Theile). Theil I: Die Grundgebilde der Ebene. Hannover, Hahn, 1898. 8vo. 8 and 95 pp. M. 3.00

LAUGEL (L.). See RIEMANN (B.).

LEVY (J.). Beiträge zur Theorie der linearen Differentialgleichungen mit Coefficienten, die von einem Parameter abhängen. [Dissertation.] Halle, 1898. 8vo. 58 pp.

MAROTTE (F.). Les equations différentielles linéaires et la théorie des groupes (thèse). Paris, Gauthier-Villars, 1898. 4to. 97 pp.

MÉRAY (C.). Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. Quatrième partie: applications géométriques classiques. Paris, Gauthier-Villars, 1898. 8vo. 11 and 248 pp.

MIKUTA (A.). See BUDISAVLJEVIC (E.).

MORLEY (F.). See HARKNESS (J.).

RIEMANN (B.). Oeuvres mathématiques de Riemann traduites par L. Laugel. Avec une préface de C. Hermite et un discours de F. Klein. Paris, Gauthier-Villars, 1898. 8vo. 35 and 453 pp. Fr. 14.00

RUEDA (C. J.). Tratado de las formas geometricas de primera categoria, precedido de una introduccion sobre los conceptos fundamentales de la gometria. Valencia, 1898. 4to. 7 and 292 pp. Fr. 12.00

SALIH ZÉKY EFENDI. Notation algébrique chez les Orientaux. (*Journal asiatique, janvier, 1898.*) Paris, 1898. 8vo. 20 pp.

SALMON (G.). Analytische Geometrie des Raumes. Deutsch bearbeitet von W. Fiedler, vierte verbesserte Auflage. (2 Theile.) Theil I. Leipzig, 1898. 8vo. 24 and 448 pp. M. 8.00

SCHILLING (F.). Geometrisch-analytische Theorie der symmetrischen S Functionen mit einem einfachen Nebenpunkt. Leipzig, 1898. 8vo. 93 pp.

SCHOUTE (P. H.). Quelques figures à $n + 2$ inversions dans l'espace à n dimensions. Partie II: Les cyclides cubiques et quartiques; les hypercycliques. Haarlem, 1898. 8vo. 58 pp.

SEVERINI (C.). Sulla rappresentazione analitica della funzioni reali discontinue di variabile reale: nota. Bologna, Gamberini e Parmeggiani, 1898. 8vo. 14 pp.

SMITH (W. B.). Infinitesimal analysis. Volume I; elementary: real variables. New York, The Macmillan Co., 1898. 8vo. 15 and 352 pp. Cloth. $3.25

STURM (C.). Lehrbuch der Analysis (Cours d'analyse). Uebersetzt von T. Gross. Band II. Berlin, Fischer, 1898. 8vo. 8 and 351 pp. M. 7.50

WEBER (H.). Traité d'algèbre supérieure. Traduit de l'allemand, sur la deuxième édition, par J. Griess. Paris, Gauthier-Villars, 1898. 8vo. 11 and 764 pp. Fr. 22.00

ZEEMAN (P.). Une surface minima algébrique du 20. ordre. Haarlem, 1898. 8vo. 47 pp.

II. ELEMENTARY MATHEMATICS

AMES (A. F.). See MCLELLAN (J. A.).

ARNOUX (V.). 2500 problèmes d'arithmétique et de géométrie pratique. Problèmes d'examen. Livre du maître. Paris, Larousse, 1898. 12mo. 490 pp. Fr. 3.00

BERTENSHAW (T. H.). Longmans' preparatory arithmetic to meet the requirements of the Oxford and Cambridge preliminary examinations, etc. London, Longmans, 1898. 8vo. 176 pp. 1s.

——The same with answers, 210 pp. 1s. 6d.

BOURLET (C.). Leçons de trigonométrie rectiligne. Paris, Colin et Ce., 1898. 8vo. 12 and 323 pp.

BOVIER-LAPIERRE (G.). Arithmétique appliquée, destinée au cours moyen pour la préparation aux examens du certificat d'études primaires, ou Recueil méthodique, avec solutions raisonnées, de six cent quatre-vingt-douze problèmes choisis dans ces examens. Paris, Bardoux, 1898. 8 and 280 pp. Fr. 2.50

CHAILAN (E.). Résumé d'algèbre élémentaire, à l'usage des élèves des classes de lettres et des candidats à la première partie du baccalauréat de l'enseignement secondaire classique. 3e edition. Paris, Poussielgue, 1898. 72 pp.

CHAUMEIL (J.) et MOREAU (G.). Premier livre d'arithmétique. Partie de l'élève. 857 exercices et problèmes. 7e édition. Paris, Larousse, 1898. 12mo. 142 pp. Fr. .75

——Deuxième livre d'arithmétique. Cours moyen et supérieur. 2000 exercices et problèmes. Partie de l'élève. 8e édition. Paris, Larousse, 1898. 12mo. 312 pp. Fr. 1.25

CHRYSTAL (G.). Introduction to algebra for the use of secondary schools and technical colleges; with answers to exercises. New York, The Macmillan Co., 1898. 12mo. 23 and 412 and 25 pp. Cloth. $1.25

DELAPIERRE (A.). Arithmétique et système métrique. Cours élémentaire, renfermant plus de 1600 problèmes et exercices. 5e édition. Paris, Larousse, 1898. 12mo. 144 pp. Fr. .70

FRÈRES DES ECOLES CHRETIENNES. Exercices et problèmes de géométrie (extraits du cours supérieur de géométrie). Paris, Poussielgue, 1898. 16mo. 96 pp.

FROLOV (M.). La théorie des parallèles démonstrée rigoureusement. Essai sur le livre I des éléments d'Euclide. Basel, 1895. 8vo. 48 pp.

FRONTERA (D. J.). See SONNET (H.).

GARCIA Y BARBARIN (E.). Soluciones à los problemas de aritmetica insertos en la obra "Elementos de Aritmetica." Madrid, 1898. 8vo. 96 pp. Fr. 3.50

GÉRARD (L.). See NIEWENGLOWSKI (B.).

GRÉVY (A.). Géométrie à l'usage des classes de lettres. Géométrie dans l'espace. Paris, Nony & Co. 140 pp.

MCLELLAN (J. A.), and AMES (A. F.). The primary public school arithmetic; based on McLellan and Dewey's "Psychology of number." New York, The Macmillan Co., 1898. 12mo. 10 and 253 pp. Cloth. 35c.

——Teachers' edition with answers. 83 and 26 pp. Cloth. $ .50

MOREAU (G.). See CHAUMEIL (G.).

NAPROVNIK (F.). Geometrie und geometrisches Zeichnen. Wien, 1898. 8vo. 8 and 200 pp. M. 2.00

NICHOLSON (J. W.). Five place logarithms and trigonometric tables, with the elementary principles of logarithms and examples for their use in computations. New York, Macmillan, 1898. 8vo. $1.50

NIEWENGLOWSKI (B.) et GÉRARD (L.). Cours de géométrie élémentaire, à l'usage des élèves de mathématiques élémentaires, de mathématiques spéciales, des candidats aux écoles du gouvernement et des candidats à l'agrégation. Géométrie plane. Paris, Carré et Naud, 1898. 8vo. 12 and 37 pp.

O'DEA (J. J.). The new explicit algebra in theory and practice. Part 2. London, Longmans, 1898. 8vo. 4s.

——The same, 2 parts in 1 volume. 6s.

REGGIO (G. Z.). Elementi di geometria e trigonometria ad uso degli instituti tecnici. Torino, 1898. 8vo. 375 pp. Fr. 5.00

SCHUBERT (H.). Vierstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen. Leipzig, 1898. 8vo. 128 pp. M. .80

SIMON (P.). Géométrie élémentaire, à l'usage des classes de lettres. Deuxième partie: géométrie dans l'espace. Paris, Verneau, 1898. 8vo. 147 pp.

SONNET (H.). Primeros elementos de geometria con las principales aplicaciones al dibujo lineal, al levantamiento de planos, à la agrimensura, etc. Traducidos del francès por D. J. Frontera. Sexta edición. Paris, Hachette, 1898. 16mo. 239 pp. Fr. 2.00

THIRD (J. A.). Modern geometry of the point, straight line and circle: an elementary treatise. London, Blackwood, 1898. 8vo. 236 pp. 3s.

WENTWORTH (G. A.). An advanced arithmetic for high schools, normal schools and academies. Boston, Ginn & Co., 1898. 15 and 400 pp. Half leather. $1.10

——New school algebra. Boston, Ginn & Co., 1898. 6 and 407 pp. Half leather. $1.25

YOUNG (J. W.). Notes on arithmetical theory. For pupil-teachers, candidates for the Queen's scholarship and other examinations. London, Simpkin, 1898. 8vo. 264 pp. 3s.

III. APPLIED MATHEMATICS.

ARNAL (L.). Traité de mécanique (statique; cinématique; dynamique; hydraulique; résistance des matériaux; chaudières à vapeur; moteurs à vapeur et à gaz). Tome 3: statique graphique et résistance des matériaux. Paris, Fanchon et Artus, 1898. 8vo. 487 pp. Fr. 16.00

—— et GAUTHIER (G.). The same. Tome 4: Chaudiéres à vapeur; moteurs à vapeur et à gaz. Paris, Fanchon et Artus, 1898. 8vo. 756 pp. Fr. 25.00

BAGGI (V.). See BOCCARDO (E. C.).

BOCCARDO (E. C.) e BAGGI (V.). Trattato elementare completo di geometria practica. Parte II (topografia). Torino, 1898. 8vo. pp. 41 to 72. Fr. 1.50

CHARRUIT (N.). Cours de géométrie cotée, à l'usage des candidats à l'Ecole spéciale militaire de Saint Cyr. Paris, Nony & Co., 1898. 8vo. 303 pp.

FRANKE (J. H.). Geodätische Punktkoordinirung im sphärischen Kleinsystem. Vergleichende Entwickelungen im einheitlichen Koordinatensystem der bayerischen Landesvermessung. Munich, Ackermann, 1898. 8vo. 6 and 81 pp. M. 2.40

GAUTHIER (G.). See ARNAL (L.).

HEEGAARD (P.). Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang. Copenhagen, 1898. 8vo. 104 pp. M. 2.50

HOLZMÜLLER (G.). Die Ingenieur-Mathematik in elementarer Behandlung, in 3-4 Theilen. Theil II: Das Potential und seine Anwendung auf die Theorien der Gravitation, des Magnetismus, der Electricität, der Wärme und der Hydrodynamik. Leipzig, 1898. 8vo. 17 and 440 pp. M. 6.00

KUTSCHER (H.). Geometrie, Feldmessen und Nivelliren. Zweite neubearbeitete Auflage. Berlin, 1898. 8vo. 164 Abbildungen. M. 1.40

LEROY (E.). Sur l'intégration des equations de la chaleur. Paris, 1898. 4to. 261 pp. Fr. 12.00

LONGCHAMPS (G. DE). Cours de problèmes de géométrie analytique, à l'usage des candidats à l'Ecole navale, à l'Ecole centrale et à l'Ecole polytechnique. Paris, Delagrave, 1898. 8vo. 8 and 436 pp.

LOW (D. A.). A pocket-book for mechanical engineers. New York, Longmans, Green & Co., 1898. 5 and 740 pp. Cloth. $2.50

NEUMANN (C.). Die elektrische Kräfte. Darlegung und genauere Betrachtung der von hervorragenden Physikern entwickelten mathematischen Theorien. Theil II. Ueber die von H. von Helmholtz in seinen älteren und neueren Arbeiten angestellten Untersuchungen. Leipzig, Teubner, 1898. 8vo. 37 and 462 pp. M. 14.00

NIPHER (F. E.). Electricity and magnetism; a mathematical treatise for advanced undergraduate students, 2d edition revised with additions. St. Louis, J. L. Boland Book and Stationery Co., 1898. 11 and 430 pp. Cloth. $3.25

PERRY (J.). Applied mechanics; a treatise for students who have time to work experimental, numerical and graphical exercises illustrating the subject. New York, D. Van Nostrand & Co., 1898. 12mo. 658 pp. Cloth. $2.50

WEISSTEIN (J.). Die rationelle Mechanik. Band I: Statik; Dynamik des Punktes. Wien, Braumüller, 1898 8vo. 18 and 350 pp. M. 10.00.

WIEDEMANN (G.). Die Lehre von der Elektricität. Zweite umgearbeitete und vermehrte Auflage. (In 5. Bänden) Band IV. Braunschweig, 1898. 8vo. 13 and 1238 pp. M. 32.00